6/26/2018

2017 Kansas Mathematics Standards

Flip Book

3rd Grade

This project used work created by the Departments of Education in Ohio, North Carolina, Georgia and resources created by Achieve the Core, EngageNY, Illustrative Mathematics, and NCTM.

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About the Flip Books

This project attempts to organize some of the most valuable resources that help develop the intent, understanding and implementation of the 2017 Kansas Mathematics Standards. These documents provide a starting point for teachers and administrators to begin discussion and exploration into the standards. It is not the only resource to support implementation of the 2017 Kansas Mathematics Standards.

This project is built on the previous work started in the summer of 2012 from Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the "flip books." The "flip books" are based on a model that Kansas had for earlier standards; however, this edition specifically targets the Kansas Mathematics Standards that were adopted in the summer of 2017. These flip books incorporate the resources from other state departments of education, the mathematics learning progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KSDE Mathematics website at http://community.ksde.org/Default.aspx?tabid=5646 and will continue to undergo changes periodically. When significant changes/additions are implemented, the modifications will be posted and dated.

Planning Advice - Focus on the Clusters

The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)

Not all standards should have the same instructional emphasis. Some groups of standards require a greater emphasis than others. In order to be intentional and systematic, priorities need to be set for planning, instruction, and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "while the remaining content is limited in scope." 4) a "lower" priority does not imply exclusion of content, but is usually intended to be taught in conjunction with or in support of one of the major clusters.

"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path, if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In order to accomplish this, educators need to think about "grain size" when planning instruction. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Phil Daro (*Teaching Chapters, Not Lessons—Grain Size of Mathematics*), strands are too vague and too large a grain size, while lessons are too small a grain size. Units or chapters produce about the right "grain size". In the planning process educators should attend to the clusters, and think of the standards as the ingredients of a cluster. Coherence of mathematical ideas and concepts exists at the cluster level across grades.

A caution--Grain size is important but can result in conversations that do not advance the intent of this structure. Extended discussions among teachers where it is argued for "2 days" instead of "3 days" on a topic because it is a lower priority can detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, focusing on lessons can provide too narrow a view which compromises the coherence value of closely related standards.

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics presents Phil Daro further explaining grain size and the importance of it in the planning process. (Click on photo to view video.)

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they determine distribution of time for both planning and instruction, helping to assure that students really understand mathematics before moving on. Each cluster has been given a priority level. As professional educators begin planning, developing and writing units, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level influenced by the work of Jason Zimba. The three levels are referred to as — **Major, Supporting** and **Additional**. Zimba suggests that about 70% of instruction should relate to the **Major** clusters. The lower two priorities (**Supporting** and **Additional**) can work together by supporting the **Major** priorities. You can find the grade Level Focus Documents for the 2017 Kansas Math Standards at: http://community.ksde.org/Default.aspx?tabid=6340.

Recommendations for Cluster Level Priorities

Appropriate Use:

- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis.
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possible quality; the additional work of the grade should support the major priorities and not detract from them.
- Set priorities for other implementation efforts such as staff development, new curriculum development, and revision of existing formative or summative testing at the state, district or school level.

Things to Avoid:

- Neglecting any of the material in the standards. Seeing Supporting and Additional clusters as optional.
- Sorting clusters (from Major to Supporting to Additional) and then teaching the clusters in order. This would remove the coherence of mathematical ideas and create missed opportunities to enhance the major work of the grade with the other clusters.
- Using the cluster headings as a replacement for the actual standards. All features of the standards matter—from the practices to surrounding text, including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise the coherence of the standards (grain size).

Mathematics Teaching Practices

(High Leverage Teacher Actions)

[National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: National Council of Teachers of Mathematics.]

The eight Mathematics Teaching Practices should be the foundation for mathematics instruction and learning. This framework was informed by over twenty years of research and presented in *Principles to Actions* by the National Council of Teachers of Mathematics (NCTM). If teachers are guided by this framework, they can move “toward improved instructional practice” and support “one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students” (NCTM, 2014, p. 12).

1. **Establish mathematics goals to focus learning.**
   Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2. **Implement tasks that promote reasoning and problem solving.**
   Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3. **Use and connect mathematical representations.**
   Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4. **Facilitate meaningful mathematical discourse.**
   Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
5. **Pose purposeful questions.**
   Effective teaching of mathematics uses purposeful questions to assess and advance students’ reasoning and sense making about important mathematical ideas and relationships.
6. **Build procedural fluency from conceptual understanding.**
   Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7. **Support productive struggle in learning mathematics.**
   Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8. **Elicit and use evidence of student thinking.**
   Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Standards for Mathematical Practice in Third Grade

The Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students grades K-12. Below are a few examples of how these Practices may be integrated into tasks that third grade students complete.

Practice	Explanation and Example
1) Make Sense and Persevere in Solving Problems.	In third grade, mathematically proficient students know that doing mathematics involves solving problems and discussing how they solved them. Students explain to themselves the meaning of a problem and look for ways to solve it. They examine problems, can make sense of the meaning of the task, and find an entry point or a way to start the task. Third grade students may use concrete objects or pictures to help them conceptualize and solve problems. They may check their thinking by asking themselves, "Does this make sense?" Students listen to other students' strategies and are able to make connections between various methods for a given problem. Example: to solve a problem involving multi-digit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach.
2) Reason abstractly and quantitatively.	Mathematically proficient students in Grade 3 recognize that a number represents a specific quantity. They connect the quantity to written symbols and create a logical representation of the problem at hand, considering both the appropriate units involved and the meaning of the quantities. This involves two processes: decontextualizing and contextualizing. In Grade 3, students represent situations by decontextualizing tasks into numbers and symbols. For example, to find the area of the floor of a rectangular room that measures 10 ft. by 12 ft., a student might represent the problem as an equation, solve it mentally, and record the problem and solution as 10 x 12 = 120 ft. squared. She has decontextualized the problem. When she states at the end that the area of the room is 120 square feet, she has contextualized the answer in order to solve the original problem. Problems like this that begin with a context and are then represented with mathematical objects or symbols are also examples of modeling with mathematics (SMP 4).
3) Construct viable arguments and critique the reasoning of others.	Mathematically proficient students in Grade 3 accurately use definitions and previously established solutions to construct viable arguments about mathematics. Grade 3 students might construct arguments using concrete referents such as objects, pictures, and drawings. They refine their mathematical communication skills as they participate in mathematical discussions involving questions like "How did you get that?" and "Why is that true?" They explain their thinking to others and respond to others' thinking. For example, when comparing the unit fractions $\frac{1}{3} and \frac{1}{6}$ students may generate their own representation of both fractions and then critique others' reasoning in class, as they connect their arguments to the representations that they created. Students in Grade 3 present their arguments in the form of representations, actions on those representations, and explanations in words (oral and written).

4) Model with mathematics.	Mathematically proficient students experiment with representing problem situations in multiple ways including numbers, words (mathematical language), drawing pictures, using objects, acting out, making a chart, list, or graph, creating equations, etc. Students require extensive opportunities to generate various mathematical representations to both equations and story problems, and explain connections between representations as well as between representations and equations. Students should be able to use all of these representations as needed. They should evaluate their results in the context of the situation and reflect on whether the results make sense.
5) Use appropriate tools strategically.	Mathematically proficient students in Grade 3 consider the available tools (including estimation) when solving a mathematical problem and decide when certain tools might be helpful. The tools that students in Grade 3 might use are: physical objects (place value (base ten) blocks, hundreds charts, number lines, tape diagrams, fraction bars, arrays or area models, tables, graphs, and concrete geometric shapes (e.g. pattern blocks, 3-D solids) paper and pencil, rulers and other measuring tools, grid paper, virtual manipulatives, and concrete geometric shapes (e.g., pattern blocks, 3-D solids), etc. Students should also have experiences with educational technologies, such as calculators and virtual manipulatives that support conceptual understanding and higher-order thinking skills. During classroom instruction, students should have access to various mathematical tools as well as paper, and determine which tools are the most appropriate to use. For example, when comparing $\frac{1}{2}\ and\ \frac{3}{4}$, students can use benchmark fractions and the number line to reason and explain that $\frac{3}{4}$ would be placed to the right of $\frac{1}{2}$ because it is a “a little more than $\frac{1}{2}$ or they might say “$\frac{3}{4}\ is\ \frac{1}{4}$ away from 1 whole”. When students model situations with mathematics, they are choosing tools appropriately (SMP 5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (SMP2).
6) Attend to precision.	Mathematically proficient students in Grade 3 are precise in their communication, calculations, and measurements. In all mathematical tasks, they communicate clearly, using grade-level appropriate vocabulary accurately as well as giving precise explanations and reasoning regarding their process of finding solutions. For example, while measuring objects iteratively (repetitively), students check to make sure that there are no gaps or overlaps. In using representations, such as pictures, tables, graphs, or diagrams, they use appropriate labels to communicate the meaning of their representation. They are careful about specifying units of measure and state the meaning of the symbols they choose. For instance, when figuring out the area of a rectangle they record their answers in square units. During tasks involving number sense, students check their work to ensure the accuracy and reasonableness of solutions.
7) Look for and make use of structure.	Mathematically proficient students in Grade 3 carefully look for patterns and structures in the number system and other areas of mathematics. Grade 3 students use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, the less you subtract, the greater the difference), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (SMP 8). For example, when Grade 3 students calculate 16 x 9, they might apply the structure of place value and the distributive property to find the product: $16 \text{ x } 9 = (10 + 6) \text{ x } 9 = (10 \text{ x } 9) + (6 \text{ x } 9)$. Students in Grade 3 should be using and explaining how they are using the different properties of operations to solve problems.
8) Look for and express regularity in repeated reasoning.	Mathematically proficient students in third grade should notice repetitive actions in computation and look for more shortcut methods. For example, students might use the distributive property as a strategy for using products they know to solve products that they don’t know. Additionally, if students are asked to find the product of 7 x 8, they might decompose 7 into 5 and 2 and then multiply 5 x 8 and 2 x 8 to arrive at 40 + 16 or 56. Third graders should continually evaluate their work by asking themselves, “Does this make sense?”

Implementing Standards for Mathematical Practice

This guide was created to help educators implement these standards into their classroom instruction. These are the practices for the **students,** and the teacher can assist students in using them efficiently and effectively.

#1 – Make sense of problems and persevere in solving them.

Summary of this Practice:

- Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, "Does this make sense?"
- Understand various approaches to solutions.

Student Actions	Teacher Actions
• Actively engage in solving problems and thinking is visible (doing mathematics vs. following steps or procedures with no understanding). • Relate current "situation" to concepts or skills previously learned, and checking answers using different methods. • Monitor and evaluate their own progress and change course when necessary. • Always ask, "Does this make sense?" as they are solving problems.	• Allow students time to initiate a plan; using question prompts as needed to assist students in developing a pathway. • Constantly ask students if their plans and solutions make sense. • Question students to see connections to previous solution attempts and/or tasks to make sense of the current problem. • Consistently ask students to defend and justify their solution(s) by comparing solution paths.

What questions develop this Practice?

- How would you describe the problem in your own words? How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem? Describe the relationship between the quantities.
- Describe what you have already tried. What might you change? Talk me through the steps you've used to this point.
- What steps in the process are you most confident about? What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

What are the characteristics of a good math task for this Practice?

- Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available.
- Encourages multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

#2 – Reason abstractly and quantitatively.

Summary of this Practice:

- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attend to the meaning of quantities, not just how to compute them.

Student Actions	Teacher Actions
• Use varied representations and approaches when solving problems. • Represent situations symbolically and manipulating those symbols easily. • Give meaning to quantities (not just computing them) and making sense of the relationships within problems.	• Ask students to explain the meaning of the symbols in the problem and in their solution. • Expect students to give meaning to all quantities in the task. • Question students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

What questions develop this Practice?

- What do the numbers used in the problem represent? What is the relationship of the quantities?
- How is_____related to______?
- What is the relationship between _____ and _____?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might you use to find a solution?
- How did you decide that you needed to use _____? Could we have used another operation or property to solve this task? Why or why not?

What are the characteristics of a good math task for this Practice?

- Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

#3 – Construct viable arguments and critique the reasoning of others.

Summary of this Practice:

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

Student Actions	Teacher Actions
• Make conjectures and exploring the truth of those conjectures. • Recognize and use counter examples. • Justify and defend all conclusions and using data within those conclusions. • Recognize and explain flaws in arguments, which may need to be demonstrated using objects, pictures, diagrams, or actions.	• Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems etc.), to support their reasoning. • Question students so they can tell the difference between assumptions and logical conjectures. • Ask questions that require students to justify their solution and their solution pathway. • Prompt students to respectfully evaluate peer arguments when solutions are shared. • Ask students to compare and contrast various solution methods • Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.)

What questions develop this Practice?

- What mathematical evidence would support your solution? How can we be sure that...? How could you prove that...?
- Will it still work if...?
- What were you considering when...? How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn’t it work? Would it ever work? Why or why not?
- What is the same and what is different about...? How could you demonstrate a counter-example?

What are the characteristics of a good math task for this Practice?

- Structured to bring out multiple representations, approaches, or error analysis.
- Embeds discussion and communication of reasoning and justification with others.
- Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- Expects students to give feedback and ask questions of others’ solutions.

#4 – Model with mathematics.

Summary of this Practice:

- Understand reasoning quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the math they know to solve problems in everyday life.
- Simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, "How can I represent this mathematically?"

Student Actions	Teacher Actions
• Apply mathematics to everyday life. • Write equations to describe situations. • Illustrate mathematical relationships using diagrams, data displays, and/or formulas. • Identify important quantities and analyzing relationships to draw conclusions.	• Demonstrate and provide students experiences with the use of various mathematical models. • Question students to justify their choice of model and the thinking behind the model. • Ask students about the appropriateness of the model chosen. • Assist students in seeing and making connections among models.

What questions develop this Practice?

- What number model could you construct to represent the problem?
- How can you represent the quantities?
- What is an equation or expression that matches the diagram..., number line…, chart..., table…?
- Where did you see one of the quantities in the task in your equation or expression?
- What math do you know that you could use to represent this situation?
- What assumptions do you have to make to solve the problem?
- What formula might apply in this situation?

What are the characteristics of a good math task for this Practice?

- Structures represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.

#5 – Use appropriate tools strategically.

Summary of this Practice:

- Use available tools recognizing the strengths and limitations of each.
- Use estimation and other mathematical knowledge to detect possible errors.
- Identify relevant external mathematical resources to pose and solve problems.
- Use technological tools to deepen their understanding of mathematics.
- Use mathematical models for visualize and analyze information

Student Actions	Teacher Actions
• Choose tools that are appropriate for the task. • Know when to use estimates and exact answers. • Use tools to pose or solve problems to be most effective and efficient.	• Demonstrate and provide students experiences with the use of various math tools. A variety of tools are within the environment and readily available. • Question students as to why they chose the tools they used to solve the problem. • Consistently model how and when to estimate effectively, and requiring students to use estimation strategies in a variety of situations. • Ask student to explain their mathematical thinking with the chosen tool. • Ask students to explore other options when some tools are not available.

What questions develop this practice?

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem? What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...?
- What can using a_____show us that _____may not?
- In what situations might it be more informative or helpful to use...?

What are the characteristics of a good math task for this Practice?

- Lends itself to multiple learning tools. (Tools may include; concrete models, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.)
- Requires students to determine and use appropriate tools to solve problems.
- Asks students to estimate in a variety of situations:
  - a task when there is no need to have an exact answer
  - a task when there is not enough information to get an exact answer
  - a task to check if the answer from a calculation is reasonable

#6 – Attend to precision.

Summary of this Practice:

- Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.
- Understand meanings of symbols used in mathematics and can label quantities appropriately.
- Express numerical answers with a degree of precision appropriate for the problem context.
- Calculate efficiently and accurately.

Student Actions	Teacher Actions
• Use mathematical terms, both orally and in written form, appropriately. • Use and understanding the meanings of math symbols that are used in tasks. • Calculate accurately and efficiently. • Understand the importance of the unit in quantities.	• Consistently use and model correct content terminology. • Expect students to use precise mathematical vocabulary during mathematical conversations. • Question students to identify symbols, quantities and units in a clear manner.

What questions develop this Practice?

- What mathematical terms apply in this situation? How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain...?
- How could you test your solution to see if it answers the problem?

What are the characteristics of a good math task for this Practice?

- Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- Expects students to use symbols appropriately.
- Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

#7 – Look for and make use of structure.

Summary of this Practice:

- Apply general mathematical rules to specific situations.
- Look for the overall structure and patterns in mathematics.
- See complicated things as single objects or as being composed of several objects.

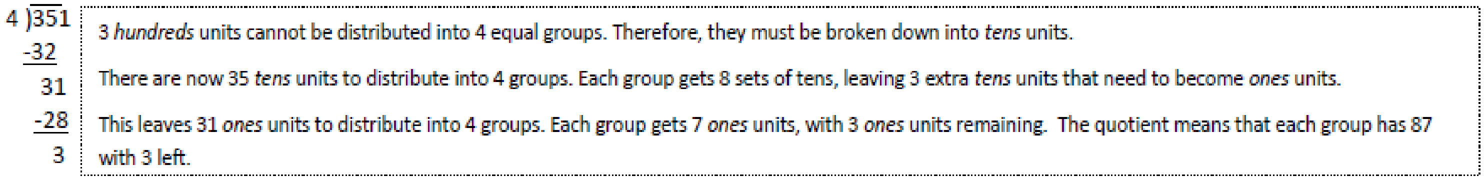
Student Actions	Teacher Actions
• Look closely at patterns in numbers and their relationships to solve problems. • Associate patterns with the properties of operations and their relationships. • Compose and decompose numbers and number sentences/expressions.	• Encourage students to look for something they recognize and having students apply the information in identifying solution paths (i.e. compose/decompose numbers and geometric figures, identify properties, operations, etc.) • Expect students to explain the overall structure of the problem and the big math idea used to solve the problem.

What questions develop this Practice?

- What observations do you make about...? What do you notice when...?
- What parts of the problem might you eliminate..., simplify...?
- What patterns do you find in...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one? How does this relate to...?
- In what ways does this problem connect to other mathematical concepts?

What are the characteristics of a good math task for this Practice?

- Requires students to look for the structure within mathematics in order to solve the problem. (i.e. – decomposing numbers by place value; working with properties; etc.)
- Asks students to take a complex idea and then identify and use the component parts to solve problems. i.e. Building on the structure of equal sharing, students connect the understanding to the traditional division algorithm. When "unit size" cannot be equally distributed, it is necessary to break down into a smaller "unit size". (example below)



4)351
-32
31
-28
3

3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units.

There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units.

This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

- Expects students to recognize and identify structures from previous experience(s) and apply this understanding in a new situation. i.e. $7 \times 8 = (7 \times 5) + (7 \times 3)$ OR $7 \times 8 = (7 \times 4) + (7 \times 4)$ new situations could be, distributive property, area of composite figures, multiplication fact strategies.

#8 – Look for and express regularity in repeated reasoning.

Summary of this Practice:

- See repeated calculations and look for generalizations and shortcuts.
- See the overall process of the problem and still attend to the details.
- Understand the broader application of patterns and see the structure in similar situations.
- Continually evaluate the reasonableness of their intermediate results.

Student Actions	Teacher Actions
• Notice if processes are repeated and look for both general methods and shortcuts. • Evaluate the reasonableness of intermediate results while solving. • Make generalizations based on discoveries and constructing formulas when appropriate.	• Ask what math relationships or patterns can be used to assist in making sense of the problem. • Ask for predictions about solutions at midpoints throughout the solution process. • Question students to assist them in creating generalizations based on repetition in thinking and procedures.

What questions develop this Practice?

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true? How would we prove that...?
- What do you notice about...?
- What is happening in this situation? What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

What are the characteristics of a good math task for this Practice?

- Present several opportunities to reveal patterns or repetition in thinking, so students can make a generalization or rule.
- Requires students to see patterns or relationships in order to develop a mathematical rule.
- Expects students to discover the underlying structure of the problem and come to a generalization.
- Connects to a previous task to extend learning of a mathematical concept.

Critical Areas for Mathematics in 3rd Grade

In Grade 3, instructional time should focus on **three** critical areas:

1. **Developing an understanding of all operations with a focus on multiplication and division and strategies for multiplication and division within 100.**
   Students develop and refine their understanding of all operations to solve multistep problems and focus on the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations (e.g., Associative Property and Distributive Property) to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division. Students understand that a word problem can be represented with an equation based on the situation, but the solution may use a related equation that is easier to manipulate (e.g., a word problem may be represented with a situation equation such as $54+?=78$; and students understand that even though the word problem is a joining situation, it is easier to solve using a subtraction equation $\{78-54=?\}$).

2. **Developing understanding of fractions, especially unit fractions (fractions with numerator of 1).**
   Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example, $\frac{1}{2}$ of the paint in a small bucket could be less paint than $\frac{1}{3}$ of the paint in a larger bucket, but $\frac{1}{3}$ of a ribbon is longer than $\frac{1}{5}$ of the same ribbon because when the ribbon is divided into 3 equal parts, the parts are longer than when the ribbon is divided into 5 equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.

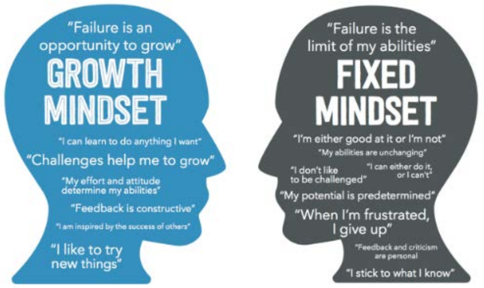
3. **Developing understanding of the structure of rectangular arrays and of area.**
   Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.

Dynamic Learning Maps (DLM) and Essential Elements

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Common Core State Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Common Core State Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

Growth Mindset



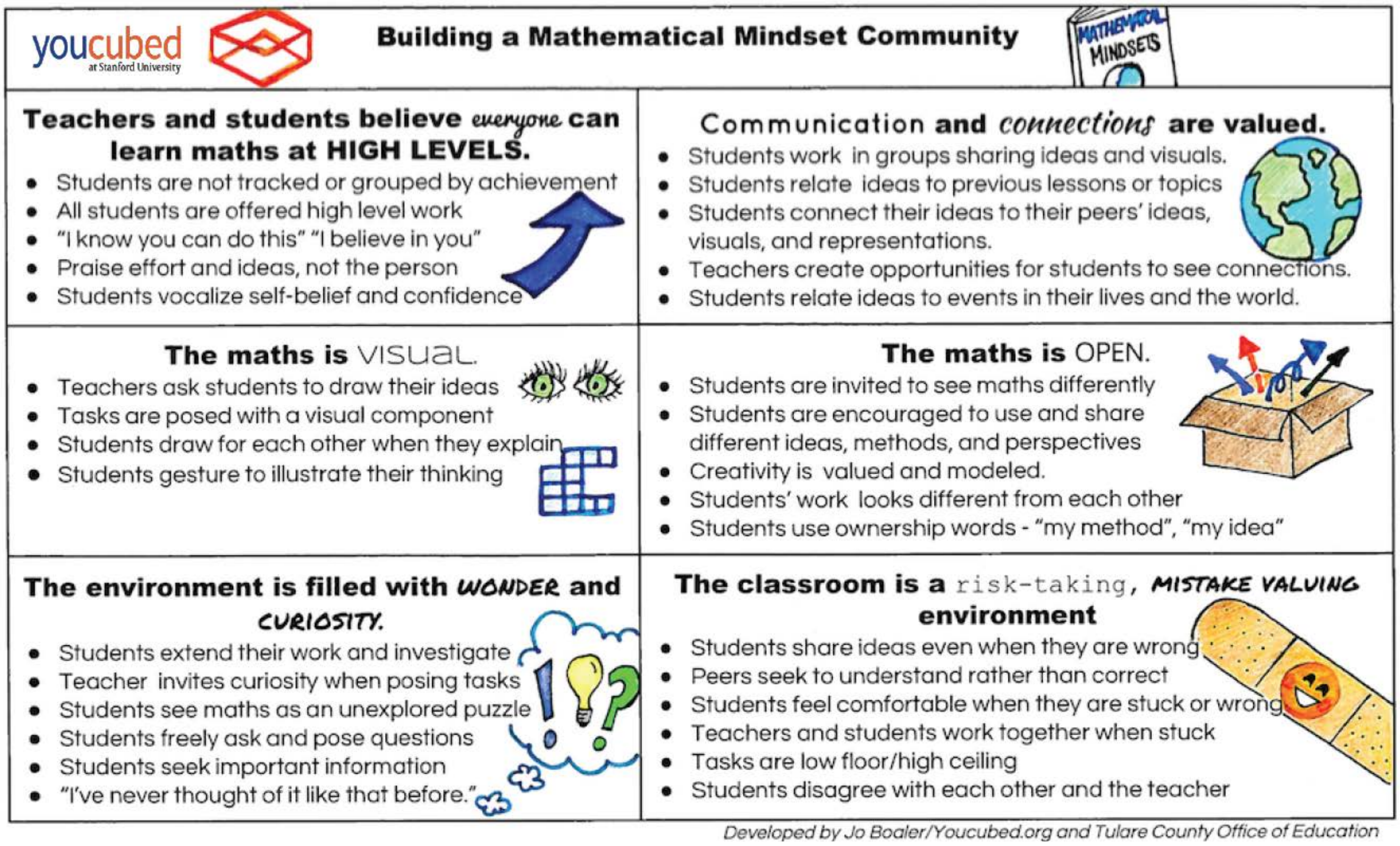
The term "growth mindset" comes from the groundbreaking work of Carol Dweck. She identified that everyone holds ideas about their own potential. Some people believe that their intelligence is more or less fixed in math – that you can do math or you can't, while others believe they can learn anything and that their intelligence can grow.

In a fixed mindset, people believe their basic qualities, like their intelligence or talent, are simply fixed traits. They spend their time documenting their intelligence or talent instead of developing it. They also believe that talent alone creates success—without effort. Students with a fixed mindset are those who are more likely to give up easily.

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Students with a growth mindset are those who keep going even when work is hard, and who are persistent.

It is possible to change mindsets and to shift students' mindsets from fixed to growth and cause higher mathematics achievement and success in life. Watch this short video to get a better understanding of what Growth Mindset is and the benefits it can bring our students.

You can find a variety of resources related to **Growth Mindset** at: http://community.ksde.org/Default.aspx?tabid=6383.



youcubed at Stanford University

Building a Mathematical Mindset Community

Teachers and students believe *everyone* can learn maths at HIGH LEVELS.

- Students are not tracked or grouped by achievement
- All students are offered high level work
- "I know you can do this" "I believe in you"
- Praise effort and ideas, not the person
- Students vocalize self-belief and confidence

Communication and *connections* are valued.

- Students work in groups sharing ideas and visuals.
- Students relate ideas to previous lessons or topics
- Students connect their ideas to their peers' ideas, visuals, and representations.
- Teachers create opportunities for students to see connections.
- Students relate ideas to events in their lives and the world.

The maths is VISUAL.

- Teachers ask students to draw their ideas
- Tasks are posed with a visual component
- Students draw for each other when they explain
- Students gesture to illustrate their thinking

The maths is OPEN.

- Students are invited to see maths differently
- Students are encouraged to use and share different ideas, methods, and perspectives
- Creativity is valued and modeled.
- Students' work looks different from each other
- Students use ownership words - "my method", "my idea"

The environment is filled with *WONDER* and *CURIOSITY.*

- Students extend their work and investigate
- Teacher invites curiosity when posing tasks
- Students see maths as an unexplored puzzle
- Students freely ask and pose questions
- Students seek important information
- "I've never thought of it like that before."

The classroom is a risk-taking, *MISTAKE VALUING* environment

- Students share ideas even when they are wrong
- Peers seek to understand rather than correct
- Students feel comfortable when they are stuck or wrong
- Teachers and students work together when stuck
- Tasks are low floor/high ceiling
- Students disagree with each other and the teacher

Developed by Jo Boaler/Youcubed.org and Tulare County Office of Education

Grade 3 Content Standards Overview

Operations and Algebraic Thinking (3.OA)

A. Represents and solves problems involving multiplication and division
   OA.1 OA.2 OA.3 OA.4

B. Understand properties of multiplication and the relationship between multiplication and division
   OA.5 OA.6

C. Multiply and divide within 100
   OA.7

D. Solve problems involving the four operations, and identify and explain patterns in arithmetic.
   OA.8 OA.9

Number and Operations in Base Ten (3.NBT)

A. Use place value understanding and properties of operations to perform multi-digit arithmetic.
   NBT.1 NBT.2 NBT.3

Number and Operations – Fractions (3.NF)

A. Develop understanding of fractions as numbers.
   NF.1 NF.2 NF.3

Measurement and Data (3.MD)

A. Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
   MD.1 MD.2 MD.3

B. Represent and interpret data.
   MD.4 MD.5

C. Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
   MD.6 MD.7 MD.8

D. Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.
   MD.9

Geometry (3.G)

A. Reason with shapes and their attributes
   G.1 G.2

Standards for Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

▶ Major Clusters ◆ Supporting Clusters ● Additional Clusters

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster A:*** *Represent and solve problems involving multiplication and division.*

Standard: 3.OA.1

Interpret products of whole numbers, *(e.g. interpret* $5 \cdot 7$ *as the total number of objects in 5 groups of 7 objects each*.) **(3.OA.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: (3.OA.1 through 3.OA.4)

This cluster is connected to:

- Connect this domain with *understanding properties of multiplication and the relationship between multiplication and division*. (Grade 3.OA.5 through 3.OA.6)
- The *use of a symbol for an unknown* is foundational for letter variables in Grade 4 when *representing problems using equations with a letter standing for the unknown quantity* (Grade 4.OA.2 and OA.3).

Explanation and Examples:

The standard interprets products of whole numbers. Students need to recognize multiplication as a means of determining the total number of objects when there are a specific number of groups with the same number of objects in each group. Multiplication requires students to think in terms of *groups of things* rather than *individual things*. At this level, multiplication is seen as “groups of” and problems such as 5 x 7 refer to 5 **groups of** 7.

It is important for teachers to understand there are several ways in which to think of multiplication:

- Multiplication is often thought of as repeated addition of equal groups. While this definition works for some sets of numbers, it is not particularly intuitive or meaningful when we think of multiplying 3 by $\frac{1}{2}$ or 5 by -2. In such cases, it may be helpful to widen the idea of grouping to include evaluation of part of a group. This concept is related to **partitioning** (which, in turn, is related to division).

  Example: Three groups of five students can be read as 3 • 5, or 15 students, while half a group of 10 stars can be represented as $\frac{1}{2} \cdot 10$, or 5 stars. These are examples of partitioning; each one of the three groups of five is part of the group of 15, and the group of 5 stars is part of the group of 10. Multiplication with fractions is NOT expected in 3rd grade but could arise in classroom discussions when working on multiplication.

- A second concept of multiplication is that of **rate or price**. Ex: If a car travels four hours at 50 miles per hour, then it travels a total of 4 • 50, or 200 miles; if CDs cost eight dollars each, then three CDs will cost 3 • $8, or $24.

- A third concept of multiplication is that of **multiplicative comparison**. Ex: Sara has four CDs, Joanne has three times as many as Sara, and Sylvia has half as many as Sara. Thus, Joanne has $3 \cdot 4$, or 12 CDs, and Sylvia has $\frac{1}{2} \cdot 4$, or 2 CDs. Again, multiplication is not expected in 3rd grade but could arise in classroom discussions.

Example for 3.OA.1:
Jim purchased 5 packages of muffins. Each package contained 3 muffins. How many muffins did Jim purchase? 5 groups of 3 OR 5 x 3 = 15.
Describe another situation where there would be 5 groups of 3.

Students recognize multiplication as a means to determine the total number of objects when there are a specific number of groups with the same number of objects in each group. Multiplication requires students to think in terms of **groups of things** rather than individual things. Students learn that the multiplication symbol '**x**' means "**groups of**" and problems such as 5 x 7 refer to 5 **groups of** 7.

To further develop this understanding, students interpret a problem situation requiring multiplication using pictures, objects, words, numbers, and equations. Then, given a multiplication expression (e.g., 5 x 6) students interpret the expression using a multiplication context. (See Table 2 in Appendix) They should begin to use the terms, *factor* and *product,* as they describe multiplication. (MP6)

Instructional Strategies: (3.OA.1 through 3.OA.4)

In Grade 2, students found the total number of objects using rectangular arrays, such as a 5 x 5, and wrote equations to represent the sum. This strategy is foundational for multiplication because students should make a connection between repeated addition and multiplication.

Students need to experience problems involving equal groups (*whole unknown* or *size of group is unknown*) and multiplicative comparison (*unknown product*, *group size unknown* or *number of groups unknown*) as shown in Table 2 in the Appendix.

Student should be encouraged to solve these problems in different ways to show the same idea and be able to explain their thinking verbally and in written form. Allowing students to present several different strategies provides the opportunity for them to compare strategies.

Provide a variety of contexts and tasks so that students will have ample opportunity to develop and use thinking strategies to support and reinforce learning of basic multiplication and division facts.

Ask students to create multiplication problem situations in which they interpret the product of whole numbers as the total number of objects in a group. Ask them to write a number model or number sentence. Also, have students create division-problem situations in which they interpret the quotient of whole numbers as the number of shares.

Students can use **known** multiplication facts to determine the **unknown fact** in a multiplication or division problem. Have them write a multiplication or division equation and the related multiplication or division equation. For example, to determine the *unknown whole* number in 27 ÷ ⍰ = 3, students should use knowledge of the related multiplication fact of 3 × 9 = 27. They should ask themselves questions such as, "How many 3s are in 27?" Have them justify their thinking with models or drawings.

Resources/Tools:

For detailed information see Operations and Algebraic Thinking Learning Progressions.

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.OA.1 to access resources specifically for this standard.

See EngageNY Modules

"Barnyard Legs", Georgia Department of Education. Students solve multiplication problems using different strategies based on Amanda Bean's Amazing Dream, A Mathematical Story by Cindy Neuschwander or a similar book about multiplication.

Georgia Department of Education

- "Twenty-Four Kids All in a Row".

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- "Exploring Equal Sets" – This four-part lesson encourages students to explore models for multiplication, the inverse of multiplication, and representing multiplication facts in equation form.
- "All About Multiplication", In this four-lesson unit, students explore several meanings and representation of multiplications and learn about properties of operations for multiplication.

Sets of counters; Number lines to skip count and relate to multiplication

Common multiplication and division situations - See Appendix, Table 2

Common Misconceptions: (3.OA.1 through 3.OA.4)

Students can overgeneralize the commutative property and think that 3 ÷ 15 = 5 and 15 ÷ 3 = 5 are the same equations. The use of models is essential in helping students eliminate this misunderstanding.

Students often believe a symbol to represent a number once will represent the same quantity in the following problem. Presenting students with multiple situations in which they select a symbol and explain what it represents and then use the same symbol in another context will counter this misconception.

Domain: Operations and Algebraic Thinking (OA)

▶ *Cluster A:* *Represent and solve problems involving multiplication and division.*

Standard: 3.OA.2

Interpret whole-number quotients of whole numbers, *(e.g. interpret* 56 ÷ 8 *as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each.)* **(3.OA.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 3.OA.1

Explanation and Examples:

This standard focuses on two distinct models of division: partition models and measurement (repeated subtraction) models.

- *Partition models* focus on the question, "How many in each group?" A context for partition models would be: There are 12 cookies on the counter. If you are sharing the cookies equally among three bags, how many cookies will go in each bag?
- *Measurement (repeated subtraction) models* focus on the question, "How many groups can you make?" A context or measurement models would be: There are 12 cookies on the counter. If you put 3 cookies in each bag, how many bags will you fill?

Students need to recognize the operation of division in two different types of situations. One situation requires determining how many groups and the other situation requires sharing (determining how many in each group). Students should be exposed to appropriate terminology (quotient, dividend, divisor, and factor). (MP6)

To develop this understanding, students interpret a problem situation requiring division using pictures, objects, words, numbers, and equations. Given a division expression (e.g., 24 ÷ 6) students interpret the expression in contexts that require both interpretations of division. (See Table 2 in Appendix)

Instructional Strategies: See 3.OA.1

Students can use **known** multiplication facts to determine the **unknown fact** in a multiplication or division problem. Have them write a multiplication or division equation and the related multiplication or division equation. For example, to determine the *unknown whole* number in 27 ÷ ⍰ = 3, students should use knowledge of the related multiplication fact of 3 × 9 = 27. They should ask themselves questions such as, "How many 3s are in 27?" Have them justify their thinking with models or drawings.

Resources/Tools

Illustrative Mathematics tasks:

- 3.OA Fish Tanks
- 3.OA Markers in Boxes

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.OA.2 to access resources specifically for this standard.

Common Misconceptions: See 3.OA.1

It is easy to confuse the number of groups and the size of the groups in some division situations. When using money, students can easily become confused. It is important to emphasize that the size of the group is the number that is repeated. It is always the size of the group.

Example: *You have $15. You want to make sure you have spending money for the next 3 weeks. How much money will you need to have for each week if you split this up equally?*

The total is $15. The number of groups is 3 (to represent the 3 weeks). The quantity being solved for represents the size of each group. Each week will get the same number so that is the size of the group.

Graphic for representing multiplication or division thinking:

Group(s)	Group Size	Total

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster A:*** *Represent and solve problems involving multiplication and division.*

Standard: 3.OA.3

Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, *(e.g. by using drawings and equations with a symbol for the unknown number to represent the problem.)* Refer to shaded section of Table 2 for specific situation types. **(3.OA.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 3.OA.1

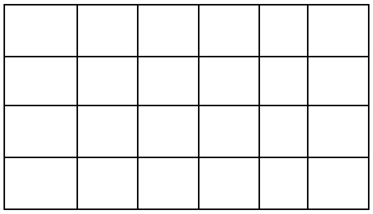
Explanation and Examples:

This standard references various strategies that can be used to solve word problems involving multiplication & division. Students should apply their skills to solve word problems. Students should use a variety of representations for creating and solving two-step word problems, such as: If you divide 4 packs of 9 brownies among 6 people, how many brownies does each person receive? *(4 x 9 = 36, 36 ÷ 6 = 6).*

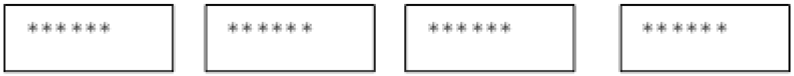
See the Appendix, Table 2, for examples of a variety of problem solving contexts, in which students need to find the product, the group size, or the number of groups. Students should be given ample experiences to explore and make sense of **ALL** the different problem structures.

Examples of Multiplication:

There are 24 desks in the classroom. If the teacher puts 6 desks in each row, how many rows are there? This task can be solved by drawing an array by putting 6 desks in each row. This is an array model.



This task can also be solved by drawing pictures of equal groups. 4 groups of 6 equals 24 objects



A student could also reason through the problem mentally or verbally, "I know 6 and 6 are 12. 12 and 12 are 24. Therefore, there are 4 groups of 6 giving a total of 24 desks in the classroom."

A number line could also be used to show jumps of equal *distance*.

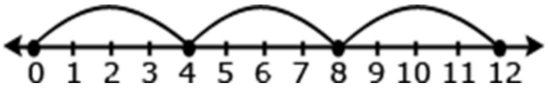
Students in third grade should use a variety of pictures and symbols to represent unknown numbers (variables). Letters are also introduced to represent unknowns in third grade.

Students use a variety of representations for creating and solving one-step word problems, i.e., numbers, words, pictures, physical objects, or equations. They use multiplication and division of whole numbers up to 10 x10. Students need to explain their thinking, show their work by using at least one representation, and verify that their answer is reasonable.
Word problems may be represented in multiple ways:

- Equations: 3 x 4 = ?, 4 x 3 = ?, ? ÷ 4 = 3, ? ÷ 3 = 4, ? = 3 x 4, ? = 4 x 3, 4 = ? ÷ 3, 3 = ? ÷ 4
- Array:

- Equal groups

- Repeated addition: 4 + 4 + 4 (or repeated subtraction for division)
- Equal jumps (distances) from 0 on the number line: 3 equal jumps to 12 or three equal jumps backwards from 12 to 0



*** As the teacher, one of the KEY understandings of multiplicative reasoning you want to develop in your students is that multiplication <u>extends beyond</u> repeated addition. If students do not move from additive to multiplicative thinking then their development in understanding higher mathematics will be compromised. Students should understand that there is a *multiplicative unit* and a *scaling factor* in the following mathematical expression – **3 x 5**. **5** is the multiplicative unit (*multiplicand*) and **3** is the scaling factor (*multiplier*) for that multiplicative unit. Essentially the expression is telling you that there are “3 groups of 5” or “3 copies of the 5”.

Examples of Division Situations:
Determining the number of objects in each share (partitive division, where the size of the groups is unknown):
There are 24 students at recess. The teacher wants to divide the class into 4 lines. Write a division equation for this story and determine how students will be in each line. *24 ÷ 4 = n. The total is known – 24 students. The number of groups is known – 4 lines. The size of the groups is unknown – how many students in each line?*

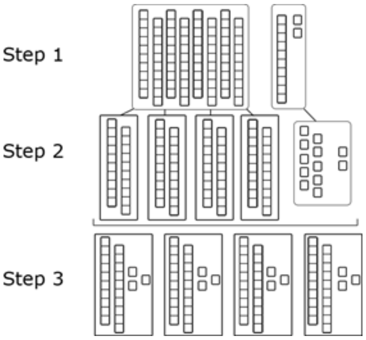
Determining the number of shares (measurement division, where the number of groups is unknown):
There are 24 students at recess. The teacher wants to divide the class into lines with 6 students in each line. Write a division equation for this story and determine how many lines the teacher will need. *24 ÷ 6 = n. The total is known – 24 students. The size of the groups is known – 6 students in each line. The number of groups is unknown – how many lines.*

Division situation with the total unknown:

There are some students at recess. The teacher wants to divide the students so that she has 4 lines with 6 students in each line. Write a division equation for this story and determine how many students the teacher will need. *n ÷ 4 = 6 OR n ÷ 6 = 4. The number of groups is known – 4 lines. The size of the groups is known – 6 students in each line. The total is unknown – how many students?* The situation is division but it is easier to solve using a multiplication equation. So the solution equation can be written as *6 x 4 = ?*

Examples of division problems using diagrams or pictures:

- Determining the number of objects in each share (partitive division, where the size of the groups is unknown):
    - The bag has 92 hair clips, and Laura and her three friends want to share them equally. How many hair clips will each person receive?



- Determining the number of shares (measurement division, where the number of groups is unknown)
    - Max the monkey loves bananas. Molly, his trainer, has 24 bananas. If she gives Max 4 bananas each day, how many days will the bananas last?

Starting	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
24	24 – 4 = 20	20 – 4 = 16	16 – 4 = 12	12 – 4 = 8	8 – 4 = 4	4 – 4 = 0

Solution: The bananas will last for 6 days.

Multiplication and Division organizational tool:

Group(s)	Group Size	Total

Instructional Strategies: See 3.OA.1

Tools/Resources:

Illustrative Mathematics tasks:

3.OA Two Interpretations of Division

3.OA Analyzing Word Problems Involving Multiplication

3.OA Gifts from Grandma, Variation 1

3.OA, MD, NBT Classroom Supplies

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.OA.3 to access resources specifically for this standard.

Thinking Blocks on Math Playground allows students several ways to model problems.

Greg Tang's Word Problem Generator – allows you to select all the various situation subtypes.
For detailed information see: Learning Progressions- Operations and Algebraic Thinking K-5

Common Misconceptions: See 3.OA.1

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster A:*** *Represent and solve problems involving multiplication and division.*

Standard: 3.OA.4

Determine the unknown whole number in a multiplication or division equation by using related equations. *For example, determine the unknown number that makes the equation true in each of the equations* $8 \cdot ? = 48$; $5 = \blacksquare \div 3$; $6\ X\ 6 = \_\_$ **(3.OA.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 3.OA.1

Explanation and Examples:

This standard refers to Table 2 in the Appendix of this document and equations for the different types of multiplication and division problem structures. The easiest problem structure includes **Unknown Product** (3 x 6 = ? or 18 ÷ 3 = 6). The more difficult problem structures include **Group Size Unknown** (3 x ? = 18 or 18 ÷ 3 = 6) or **Number of Groups Unknown** (? x 6 = 18, 18 ÷ 6 = 3).

The focus of 3.OA.4 goes beyond the traditional notion of *fact families*, by having students explore the **inverse relationship** of multiplication and division. Related equations allow students to see both composition and decomposition equations.

Students apply their understanding of the meaning of the equal sign as "the same value as" to interpret an equation with an unknown. When given 4 x *n* = 40, they might think:

- 4 groups of what size of each group is the same as 40 **OR** 4 groups of some number is the same as 40

Students apply their understanding of the meaning of the equal sign as "the same value as" to interpret an equation with an unknown. When given 4 x *n* = 40, they might think:

- 4 groups of some number is the same as 40
- I know that 4 groups of 10 is 40 so the unknown number is 10
- The missing factor is 10 because 4 times 10 equals 40.

Equations in the form of a x b = c and c = a x b should be used interchangeably, with the unknown in different positions.

Example:

Solve the equations below:

$$24 = ? \text{ x } 6$$

I have 24 total. If I have 6 groups, how many items should be in each of my groups? OR I have 24 total. If I have 6 items in each group, how many groups will I have?

$$72 \div \blacksquare = 9$$

I have 72 total. What multiplied by 9 gives me a result of 72?

Melisa has 3 bags. There are 4 marbles in each bag. How many marbles does Melisa have altogether? *3 x 4 = m*

This standard is strongly connected to 3.OA.3 when students solve problems and determine unknowns in equations.

Students should experience creating story problems for given equations. When crafting story problems, they should carefully consider the question(s) to be asked and answered to write an appropriate equation. Students may approach the same story problem differently and write either a multiplication equation or division equation. Use Table 2 in the Appendix to help students understand situation equations. The situation equation should mirror what is happening in the problem.

Instructional Strategies: See 3.OA.1

Resources/Tools:

Illustrative Mathematics tasks:

- 3.OA Finding the unknown in a division equation

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.OA.4 to access resources specifically for this standard.

Thinking Blocks on Math Playground allows students several ways to model problems.

Common Misconceptions: See 3.OA.1 & 3.OA.2

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster B**: <u>Understand properties of multiplication and the relationship between multiplication and division.</u>*

Standard: 3.OA.5

Apply properties of operations as strategies to multiply and divide. *Examples: If* $6 \cdot 4 = 24$ *is known, then* $4 \cdot 6 = 24$ *is also known. (Commutative property of multiplication.)* $3 \cdot 5 \cdot 2$ *can be found by* $3 \cdot 5 = 15$*, then* $15 \cdot 2 = 30$*, or by* $5 \cdot 2 = 10$*, then* $3 \cdot 10 = 30$*. (Associative property of multiplication.) Knowing that* $8 \cdot 5 = 40$ *and* $8 \cdot 2 = 16$*, one can find* $8 \cdot 7$ *as* $8 \cdot (5 + 2) = (8 \cdot 5) + (8 \cdot 2) = 40 + 16 = 56$*. (Distributive property.) Students need not use formal terms for these properties.* **(3.OA.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.6 Attend to precision.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: (3.OA.5 through 3.OA.6)

This cluster is connected to:

- Third grade 3.OA.A (*Represent and solve problems involving multiplication and division*).
- Second grade 2.OA.C (*Work with equal groups of objects to gain foundations for multiplication*) and 2.G.2 (*Partition a rectangle into rows and columns of same-size squares and count to find the total number of them*).

Explanation and Examples:

This standard references properties of multiplication. While students **DO NOT** need to use the formal terms of these properties, students should understand that properties are rules about how numbers work. Teachers should use the correct terminology when possible.

Students need to be flexible and fluent when applying each of the properties. Students represent expressions using various objects, pictures, words and symbols in order to develop their understanding of properties. They multiply by 1 and 0 and divide by 1. They change the order of numbers to determine that the order of numbers does not make a difference in multiplication (**but does make a difference in division**). Given three factors, they investigate how changing the order of how they multiply the numbers does not change the product. They also decompose numbers to build fluency with multiplication.

The *commutative property* (order property) states that the order of numbers does not matter when adding or multiplying numbers. For example, if a student knows that 5 x 4 = 20, then they also know that 4 x 5 = 20.

The array below could be described as a 5 x 4 array for 5 columns and 4 rows, or a 4 x 5 array for 4 rows and 5 columns. **There is no "fixed" way to write the dimensions of an array as rows x columns or columns x rows.** Students should have flexibility in being able to describe both dimensions of an array.

X X X X		**X X X X X**
X X X X	**4 x 5**	**X X X X X**
X X X X	**or**	**X X X X X**
X X X X	**5 x 4**	**X X X X X**
X X X X		

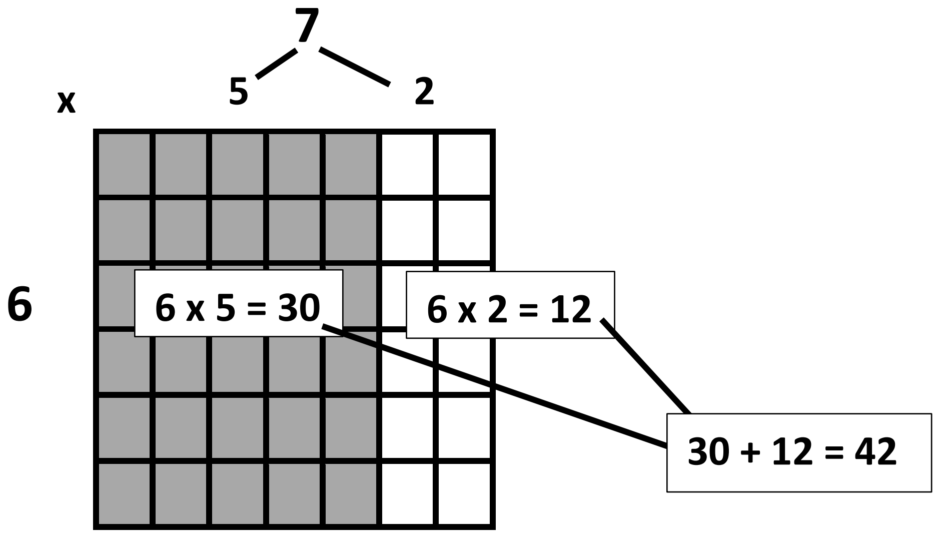
The *associative property* states that the sum or product stays the same when the grouping of addends or factors is changed. For example, when a student multiplies 7 x 5 x 2, a student could rearrange the numbers to first multiply 5 x 2 = 10 and then multiply 10 x 7 = 70.

Students should be introduced to the *distributive property* of multiplication over addition as a strategy for using products they know to solve products they don't know. Students use mental math to determine a product.

Here are some examples of how students could use the distributive property to find the product of 7 x 6. Again, students should use the distributive property, but can refer to this method using informal language such as "breaking numbers apart".

Example:
Students determine the products and factors of problems by breaking numbers apart. For example, for the problem 6 x 7 = ?, students can decompose the 7 into a 5 and 2, and reach the answer by multiplying 6 x 5 = 30 and 6 x 2 =12 and adding the two products (30+12=42).

6 x 7 = ?



Mental Math Examples:
Here are ways that students could use the distributive property to determine the product of 7 x 6. Again, students should use the distributive property, but can refer to this in informal language such as “breaking numbers apart”.

Student 1	Student 2	Student 3
7 x 6	7 x 6	7 x 6
7 x 5 = 35	7 x 3 = 21	5 x 6 = 30
7 x 1 = 7	7 x 3 = 21	2 x 6 = 12
35 + 7 = 42	21 + 21 = 42	30 + 12 = 42

To further develop understanding of properties related to multiplication and division, students use different representations and their understanding of the relationship between multiplication and division to determine if the following types of equations are **true** or **false**.

0 x 7 = 7 x 0 = 0 (Zero Property of Multiplication)
1 x 9 = 9 x 1 = 9 (Multiplicative Identity Property of 1)
3 x 6 = 6 x 3 (Commutative Property)
8 ÷ 2 ≠ 2 ÷ 8 (Students are only to determine that these are not equal)
2 x 3 x 5 = 6 x 5
10 x 2 < 5 x 2 x 2
2 x 3 x 5 = 10 x 3
1 x 6 > 3 x 0 x 2

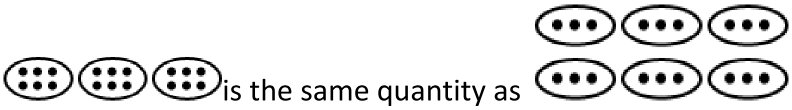
Students represent equations and inequalities using various objects, pictures, words and symbols in order to develop their understanding of properties. They multiply by 1 and 0 and divide by 1, never by 0. They change the order of numbers to determine that the order of numbers does not make a difference in multiplication (but does make a difference in division).

Given three factors, they investigate changing the order of how they multiply the numbers to determine that changing the order does not change the product. They also decompose numbers to build fluency with multiplication.

Use models to help build understanding of the *commutative property*:

Example: 3 x 6 = 6 x 3

In the following diagram it may not be obvious that 3 groups of 6 is the same as 6 groups of 3. A student may need to count to verify this.



Different representation:

An array explicitly demonstrates the concept of the commutative property. The array just needs to be rotated.

4 rows of 3 or 4 x 3

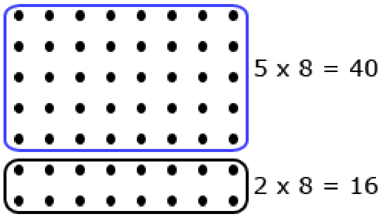
3 rows of 4 or 3 x 4

Students are introduced to the *distributive property of multiplication over addition* as a strategy for using products they know to solve products they don't know. For example, if students are asked to find the product of 7 x 8, they might decompose 7 into 5 and 2 and then multiply 5 x 8 and 2 x 8 to arrive at 40 + 16 or 56.

Students should learn that they can decompose either of the factors. It is important to note that the students may record their thinking in different ways.

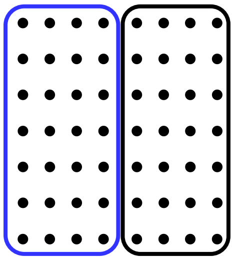
Decomposing the 7:

5 x 8 = 40
2 x 8 = 16
56



Decomposing the 8:

7 x 4 = 28
7 x 4 = 28
56



Instructional Strategies: **(3.OA.5 through 3.OA.6)**

Students need to apply properties of operations (commutative, associative and distributive) as strategies to multiply and divide. Applying the concept involved is more important than students knowing the name of the property.

Understanding the commutative property of multiplication is developed through the use of models as basic multiplication facts are learned. For example, the result of multiplying 3 x 5 (15) is the same as the result of multiplying 5 x 3 (15).

Splitting arrays can help students understand the distributive property. They can use a known fact to learn other facts that may cause difficulty. (See example above where students split an array into smaller arrays and add the sums of the groups.)

Students' understanding of the part/whole relationships is critical in understanding the connection between multiplication and division.

Resources/Tools

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- "Multiplication--It's In the Cards" – Students skip-count and examine multiplication patterns. They also explore the commutative property of multiplication.
- "Multiplication--It's In the Cards: Looking for Calculator Patterns". – Students use a web-based calculator to create and compare counting patterns using the constant function feature of the calculator. Making connections between multiple representations of counting patterns reinforces students understanding of this important idea and helps them recall these patterns as multiplication facts. From a chart, students notice that multiplication is commutative.

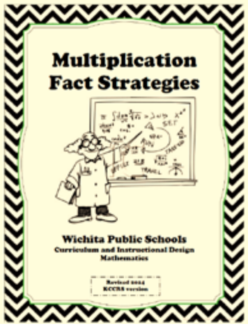
Illustrative Mathematics tasks:

- 3.OA Valid Equalities? (Part 2)

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.OA.5 to access resources specifically for this standard.

Thinking Blocks on Math Playground allows students several ways to model problems.

Access the **Multiplication Fact Strategies** book from the KSDE Mathematics website for lessons, activities, and games that center on using the properties of operations to build fact fluency:



Multiplication Fact Strategies

Wichita Public Schools
Curriculum and Instructional Design
Mathematics

Revised 2014
KCCRS version

Common Misconceptions:

Students may experience difficulty in determining which factor represents rows or the number of objects in a group, and which factor represents the number of groups or columns. In division there are two different situations that can cause confusion depending on which factor is the unknown—the number in the group (size of the group) or the number of groups.

Students will often believe that these properties hold true for division. They must be provided opportunities to see how this is not true. Telling students is not enough. They must experience problems that challenge their beliefs and come to their own conclusions.

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster B:*** *Understand properties of multiplication and the relationship between multiplication and division.*

Standard: 3.OA.6

Understand division as an unknown-factor problem. *For example, find* 32 ÷ 8 *by finding the number that makes 32 when multiplied by 8.* **(3.OA.6)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.7 Look for and make use of structure.

Connections: See 3.OA.5

Explanation and Examples:

This standard refers to some of the situations in Table 2 (see Appendix). Since multiplication and division are inverse operations, students are expected to solve problems and explain their processes of solving division problems that can also be represented as unknown factor in multiplication problems.

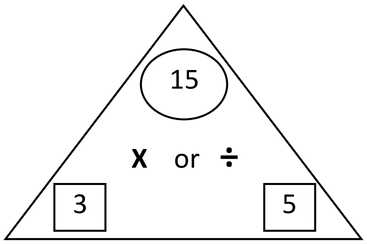
Example:

A student knows that 2 x 9 = 18. How can they use that fact to determine the answer to the following question: 18 people are divided into pairs in P.E. class? How many pairs are there? Write a division equation and explain your reasoning.

Multiplication and division are inverse operations and that understanding can be used to find the unknown. Fact triangles demonstrate the inverse operations of multiplication and division by showing the two factors and how those factors relate to the product and/or quotient.

Example:

- 3 x 5 = 15   5 x 3 = 15
- 15 ÷ 3 = 5   15 ÷ 5 = 3
- 15 = 3 x 5   15 = 5 x 3
- 5 = 15 ÷ 3   3 = 15 ÷ 5



Students use their understanding of the meaning of the equal sign as "the same value as" to interpret an equation with an unknown. When given 32 ÷ ■ = 4, students may think:

- 4 groups of some number is the same as 32.
- I know that 4 groups of 8 is 32 so the unknown number is 8.
- The missing factor is 8 because 4 times 8 is 32.

Equations in the form of a ÷ b = c and c = a ÷ b need to be used interchangeably, with the unknown in different positions.

Instructional Strategies: See 3.OA.5

Common Misconceptions: See 3.OA.5

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster C:*** *Multiply and divide within 100 (basic facts up to 10 x 10).*

Standard: 3.OA.7

Fluently (efficiently, accurately, and flexibly) multiply and divide with single digit multiplications and related divisions using strategies *(e.g. relationship between multiplication and division, doubles, double and double again, half and then double, etc.)* or properties of operations. **(3.OA.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to:

- Third Grade 3.OA.A (**Developing understanding of multiplication and division and strategies for multiplication and division within 100**) and 3.OA.B (**Understand properties of multiplication and the relationship between multiplication and division**).

Explanation and Examples:

This standard uses the word fluently, which means accuracy, efficiency (**using a reasonable amount of steps and time**), and flexibility (using strategies such as the distributive property). **"Know from memory" does not mean focusing only on timed tests and repetitive practice,** but ample experiences working with manipulatives, pictures, arrays, word problems, and numbers to internalize the basic facts (up to 9 x 9). Strategies using decomposition and the properties of multiplication will lead to fluency and better retention of facts over time.

By studying patterns and relationships in multiplication facts and relating multiplication and division, students build a foundation for fluency with multiplication and division facts. Students demonstrate fluency with multiplication facts through 10 and the related division facts. Multiplying and dividing fluently refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

Strategies students may use to attain fluency include:

- Multiplication by zeroes and ones
- Doubles (2s facts)
- Double and Double Again (4s)
- Doubling three times (8s)
- Tens facts (relating to place value, 5 x 10 is 5 tens or 50)
- Five facts (half of tens or connect to the analog clock)
- Skip counting (counting groups of __ and knowing how many groups have been counted)
- Square numbers (the physical and visual representation of these facts makes a square - ex: 3 x 3)
- Nines (10 groups less 1 group; e.g., 9 x 3 is 10 groups of 3 minus 1 group of 3 so 30 – 3 = 27)
- Decomposing into known facts (6 x 7 is a double - 6 x 6 - plus one more group of 6)
- Turn-around facts (Commutative Property)

- Related Equations *also known as fact families* (Ex: 6 x 4 = 24; 24 ÷ 6 = 4; 24 ÷ 4 = 6; 4 x 6 = 24; 24 = 6 x 4; 24 = 4 x 6; 6 = 24 ÷ 4; 4 = 24 ÷ 6)

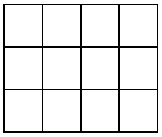
General Note: Students should have exposure to multiplication and division problems presented in both **vertical** and **horizontal** forms. *(Problems presented horizontally encourages mental computation.)*

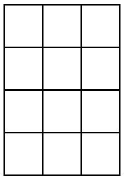
Instructional Strategies:

Students need to understand the part/whole relationships in order to understand the connection between multiplication and division. They need to develop efficient strategies that lead to the big ideas of multiplication and division.

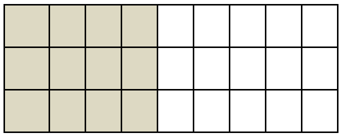
- These big ideas include understanding the **properties of operations**, such as the *commutative* and *associative properties of multiplication* and the *distributive property*. The naming of the property is not necessary at this stage of learning.
- In Grade 2, students found the total number of objects using rectangular arrays, such as a 5 x 5, and wrote equations to represent the sum. **This is called unitizing**. It requires students to count groups, not just objects. They see the whole as a number of groups of a number of objects. This strategy is a foundation for multiplication helping students make a connection between repeated addition and multiplication.

As students create arrays for multiplication using objects or drawing on graph paper, they should discover that three groups of four and four groups of three yield the same results. They should observe that the arrays contain the same total number of squares but the orientation of the array has just rotated so the rows and columns are switched. Provide numerous situations for students to develop this understanding. (*Commutative property*)





To develop an understanding of the *distributive property*, students need to decompose the whole into groups. Arrays are valuable tools and should be used to develop this understanding. To find the product of 3 × 9, students can decompose 9 into the sum of 4 and 5 and find 3 × (4 + 5).



The *distributive property* is the basis for the standard multiplication algorithm that students will use to fluently multiply multi-digit whole numbers in Grade 5.

Once students have an understanding of multiplication using efficient strategies, they should make the connection to division.

Using various strategies to solve different contextual problems that use the same two one-digit whole numbers requiring multiplication allows for students to commit to memory all products of two one-digit numbers.

Resources/Tools:

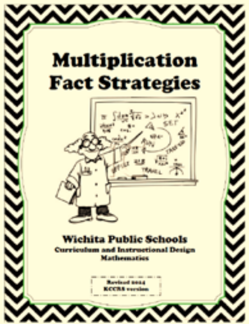
- Unifix cubes
- Grid or graph paper
- Sets of counters

See: K-5 Operations and Algebraic Thinking and Counting and Cardinality for detailed information.

Georgia Department of Education:

- "A Giraffe Named Stretch" - Students create and solve multiplication stories about Stretch (a giraffe) and his children using a list of facts given to them.
- "Making Sense of Division" - Students demonstrate how to use division as an application of money. Students observe what happens when an amount of money is divided evenly among a group of people or not divided evenly among a group of people.

Access the **Multiplication Fact Strategies** book from the KSDE Mathematics website for lessons, activities, and games that center on using the properties of operations to build fact fluency:



Multiplication
Fact Strategies

Wichita Public Schools
Curriculum and Instructional Design
Mathematics

Revised 2014
KCCRS version

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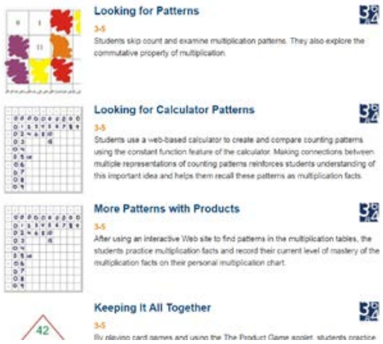
- Multiplication: It's in the Cards
  (view of some of the lessons)



Looking for Patterns
3-5
Students skip count and examine multiplication patterns. They also explore the commutative property of multiplication.

Looking for Calculator Patterns
3-5
Students use a web-based calculator to create and compare counting patterns using the constant function feature of the calculator. Making connections between multiple representations of counting patterns reinforces students understanding of this important idea and helps them recall these patterns as multiplication facts.

More Patterns with Products
3-5
After using an interactive Web site to find patterns in the multiplication tables, the students practice multiplication facts and record their current level of mastery of the multiplication facts on their personal multiplication chart.

Keeping It All Together
3-5
By playing card games and using the The Product Game applet, students practice

Common Misconceptions:

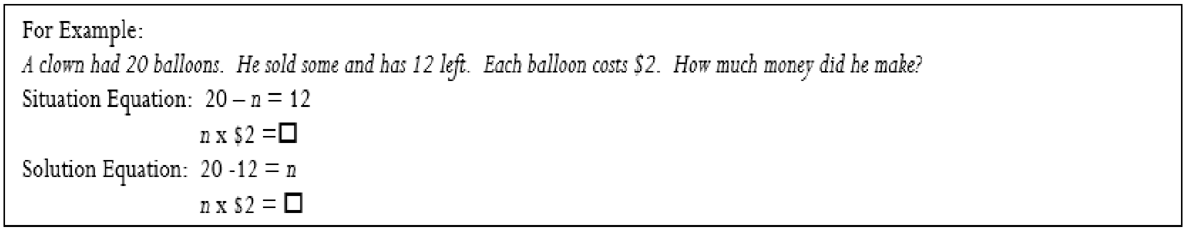
Student who struggle most likely do not have fluency for the easy numbers. The child does not understand an unknown factor (a divisor) can be found from the related multiplication. It is not a matter of instilling facts divorced from their meaning, but rather the outcome of carefully designed learning. That involves the interplay of practice and reasoning.

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster D:*** *Solve problems involving the four operations, and identify and explain patterns in arithmetic.*

Standard: 3.OA.8

Solve two-step word problems using any of the four operations. Represent these problems using both situation equations and/or solution equations with a letter or symbol standing for the unknown quantity (*refer to Table 1 and Table 2 and standard 3.OA.3*). Assess the reasonableness of answers using mental computation and estimation strategies including rounding. This standard is limited to problems posed with whole numbers and having whole-number answers. **(3.OA.8)**



For Example:

A clown had 20 balloons. He sold some and has 12 left. Each balloon costs $2. How much money did he make?

Situation Equation: 20 – n = 12
n x $2 = ☐
Solution Equation: 20 -12 = n
n x $2 = ☐

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to:

- Represent and solve problems involving multiplication and division. 3.OA.A
- Use place value understanding and properties of operations to perform multi-digit arithmetic. 3.NBT.A

Explanation and Examples:

This standard refers to two-step word problems using the four operations. The size of the numbers should be limited to 3rd grade standards (e.g., 3.OA.7 and 3.NBT.2). Adding and subtracting numbers should include numbers within 1000, and multiplying and dividing numbers should include single-digit factors and products less than 100.

This standard also expects students to represent problems using equations with a letter to represent unknown quantities.

Example:

Mike runs 2 miles a day. His goal is to run 25 miles. After 5 days, how many miles does Mike have left to run in order to meet his goal? Write an **equation** to show this problem situation. (25 = 2 x 5 + *m*).

This standard refers to estimation strategies, including rounding which would expect the use of compatible numbers (numbers that sum to 10, 50, or 100). The focus in this standard is to have students use and discuss various strategies. Students should estimate during problem solving, and then revisit their estimate to check for reasonableness.

Example:

Here are some typical estimation strategies for the following problem:

On a vacation, your family travels 267 miles on the first day, 194 miles on the second day and 34 miles on the third day. About how many total miles did they travel?

Student 1	Student 2	Student 3
I first thought about 267 and 34. I noticed that their sum is about 300. Then I knew that 194 is close to 200. When I put 300 and 200 together, I get 500.	I first thought about 194. It is really close to 200. I also have 2 hundreds in 267. That gives me a total of 4 hundreds. Then I have 67 in 267 and the 34. When I put 67 and 34 together that is really close to 100. When I add that hundred to the 4 hundreds that I already had, I end up with 500.	I rounded 267 to 300. I rounded 194 to 200. I rounded 34 to 30. When I added 300, 200 and 30, I know my answer will be about 530.

When assessing estimation strategies you could have more than one reasonable answer (500 or 530), or students could provide a range (between 500 and 550). Problems should be structured so that all acceptable estimation strategies will arrive at a reasonable answer. Students should be expected to explain their thinking in arriving at their estimation.

Student should use various estimation skills solve word problems. They should include:

- identifying when estimation is appropriate
- determining the level of accuracy needed
- selecting the appropriate method of estimation
- verifying solutions or determining the reasonableness of solutions.

Estimation strategies include, but are not limited to:

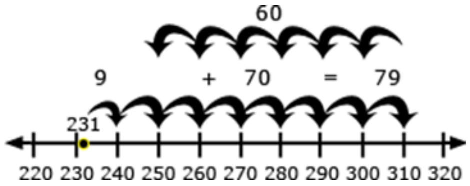
- using benchmark numbers that are easy to compute
- front-end estimation with adjusting:
  1. (using the highest place value and estimating from the front end making adjustments to the estimate by taking into account the remaining amounts)
  2. rounding and adjusting (students round down or round up and then adjust their estimate depending on how much the rounding changed the original values)

Problem Solving without Estimation

It is important that students be exposed to multiple problem-solving strategies (using any combination of words, numbers, diagrams, physical objects or symbols) and be able to choose which ones to use.

Examples:

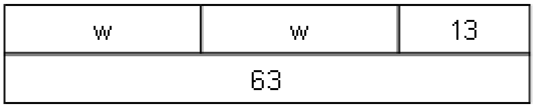
- Jerry earned 231 points at school last week. This week he earned 79 points. If he uses 60 points to earn free time on a computer, how many points will he have left?



A student may use the number line above to describe his/her thinking, "231 + 9 = 240 so now I need to add 70 more. 240, 250 (10 more), 260 (20 more), 270, 280, 290, 300, 310 (70 more). Now I need to count back 60. 310, 300 (back 10), 290 (back 20), 280, 270, 260, 250 (back 60)."

A student writes the equation, 231 + 79 – 60 = m and uses rounding (230 + 80 – 60) to estimate.
A student writes the equation, 231 + 79 – 60 = m and calculates 79-60 = 19 and then calculates 231 + 19 = m.

The soccer club is going on a trip to the water park. The cost of attending the trip is $63. Included in that price is $13 for lunch and the cost of 2 wristbands, one for the morning and one for the afternoon. Write an equation representing the cost of the field trip and determine the price of one wristband.



The above diagram helps the student write the equation, w + w + 13 = 63. Using the diagram, a student might think, "I know that the two wristbands cost $50 ($63-$13) so one wristband costs $25." To check for reasonableness, a student might use front end estimation and say 60-10 = 50 and 50 ÷ 2 = 25.

Instructional Strategies: (3.OA.8 through 3.OA.9)
Students gain a full understanding of which operation to use in any given situation through contextual problems. Number skills and concepts are developed as students solve problems. Problems should be presented on a regular basis as students work with numbers and computations.

When students think about the situation of the problem and write an equation to fit the situation, then they can use their understanding of how operations work to create an equation that will be easier to use to solve the problem. Here is a simple example: *I had 24 cookies. My friend gave me some more cookies so now I have 36. How many did my friend give me?* To tell students to subtract can be confusing to them because this is definitely a joining situation. But if students write the equation 24 + *n* = 36, then they can think about the relationship between addition and subtraction to create a related equation (solution equation) in order to find the solution. 36 – 24 = *n*.

Researchers and mathematics educators advise against providing "key words" for students to look for in problem situations because they can be misleading and if key words are helpful they only provide a clue to one of the steps in the problem. This is not helpful now that students are expected to solve multistep problems. Students should use various strategies to solve problems. Students should analyze the structure of the problem to make sense of it. They should think through the problem and the meaning of the answer before attempting to solve it.

Encourage students to represent the problem situation with a drawing or with counters/blocks. Students should determine the reasonableness of the solution to all problems using mental computations and estimation strategies.

Students can use base–ten blocks on centimeter grid paper to construct rectangular arrays to represent problems.

Students should use arithmetic patterns and explain the patterns using properties of operations. They can explore patterns by determining likenesses, differences and changes. Use patterns in addition and multiplication tables.

Resources/Tools:

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Illustrative Mathematics tasks:

- 3.OA The Stamp Collection
- 3.OA The Class Trip

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.OA.8 to access resources specifically for this standard.

Thinking Blocks on Math Playground allows students several ways to model problems.

Common Misconceptions:

Students frequently learn computation strategies without understanding the benefit of estimation. Estimation assists in determining if answers are reasonable are not. You may decide to work on estimation before tackling specific strategies for computation. Talk about the reasonableness of answers and have the students defend their estimations.

If students are not allowed to think about the situation of the problems then they begin to believe that math doesn't make sense. If the problem is a joining situation but you tell the students to subtract without making the connection to the relationship between the operations, then this will lead to confusion and misconceptions that are hard to overcome.

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster D:*** *Solve problems involving the four operations, and identify and explain patterns in arithmetic.*

Standard: 3.OA.9

Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations (See Table 5). *For example, observe that 4 times a number is always even, and explain why 4 times a number can be* ***decomposed*** *into two equal addends.* **(3.OA.9)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 3.OA.8

Explanation and Examples:

This standard calls for students to examine arithmetic patterns involving both addition and multiplication.

Arithmetic patterns are patterns that change by the same rate, such as adding the same number. For example, the series 2, 4, 6, 8, 10 is an arithmetic pattern that increases by 2 between each term.

This standard also mentions identifying patterns related to the properties of operations.

Examples:

- Even numbers are always divisible by 2. Even numbers can always be decomposed into 2 equal addends (14 = 7 + 7).
- Multiples of even numbers (2, 4, 6, and 8) are always even numbers.
- On a multiplication chart, the products in each row and column increase by the same amount (skip counting).
- On an addition chart, the sums in each row and column increase by the same amount.
- Using a multiplication table, highlight a row of numbers and ask students what they notice about the highlighted numbers.

Explain a pattern using properties of operations.
When (commutative property) one changes the order of the factors they will still get the same product, example 6 x 5 = 30 and 5 x 6 = 30.

Teacher: What pattern do you notice when 2, 4, 6, 8, or 10 are multiplied by any number (even or odd)?

Student: The product will always be an even number.

Teacher: Why?

In an addition table ask what patterns they notice. Explain why the pattern works this way?

Students need ample opportunities to observe and identify important numerical patterns related to operations. They should build on their previous experiences with properties related to addition and subtraction. Students investigate addition and multiplication tables in search of patterns and explain why these patterns make sense mathematically. (MPs 7&8).

*All of the **understandings** of multiplication and division situations, of the levels of representation and solving, and of patterns need to culminate by the end of Grade 3 in fluent multiplying and dividing of all single digit numbers and 10.*

It should be clear, this does not mean instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involved the interplay of PRACTICE and REASONING. (Learning Progressions-Operations and Algebraic Thinking K-5).

Examples:

- Any sum of two even numbers is even.
- Any sum of two odd numbers is even.
- Any sum of an even number and an odd number is odd.
- The multiples of 4, 6, 8, and 10 are all even because they can all be decomposed into two equal groups.
- The doubles (2 addends the same) in an addition table fall on a diagonal while the doubles (multiples of 2) in a multiplication table fall on horizontal and vertical lines.
- The multiples of any number fall on a horizontal and a vertical line due to the commutative property.
- All the multiples of 5 end in a 0 or 5 while all the multiples of 10 end with 0. Every other multiple of 5 is a multiple of 10.

Students also investigate a hundreds chart in search of addition and subtraction patterns. They record and organize all the different possible sums of a number and explain why the pattern makes sense.

Addend	Addend	Sum
0	20	20
1	19	20
2	18	20
3	17	20
4	16	20
?	?	?
?	?	?
?	?	?
20	0	20

Instructional Strategies: See 3.OA.8

Resources/Tools:

Illustrative Mathematics tasks:

- 3.OA Addition Patterns
- 3.OA Patterns in the multiplication table
- 3.OA Symmetry of the addition table
- 3.OA Making a ten

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.OA.9 to access resources specifically for this standard.

Common Misconceptions:

The student is not able to follow the conventions of order of operations. They randomly attack pairs of numbers without regard for what the associative and distributive properties require. They do not look for and make use of structure (MP7) or they do not follow the "rules of the road".

Domain: Number and Operations in Base Ten (NBT)

● ***Cluster A:*** *Use place value understanding and properties of operations to perform multi-digit arithmetic.*

Standard: 3.NBT.1

Use place value understanding to round whole numbers to the nearest 10 or 100. **(3.NBT.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: (3.NBT.1 through 3.NBT.3)

This cluster is connected to:

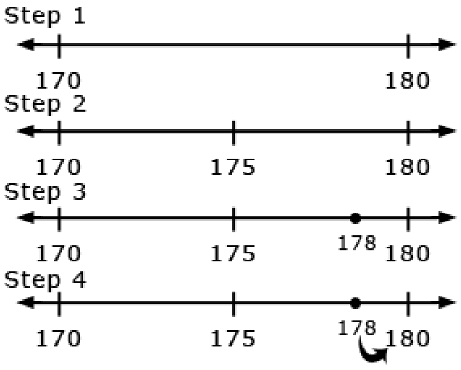
- The content in this cluster goes beyond the critical areas to address solving multi-step problems.
- The rounding strategies developed in third grade will be expanded in grade four with larger numbers.
- Additionally, students will formalize the rules for rounding numbers with the expansion of numbers in fourth grade.
- In fourth grade, the place value concepts developed in grades K-3 will be expanded to include decimal notation.
- Understand place value. (2.NBT.1 – 4 and 2.NBT.5 – 9)

Explanation and Examples:

This standard refers to place value understanding, which extends beyond an algorithm or procedure for rounding. The expectation is that students have a deep understanding of place value and number sense and can explain and reason about the answers they get when they round.

Students should have numerous experiences using a number line and a hundred chart to support their work with rounding. Students learn **when** and **why** to round numbers. They identify possible answers and halfway points. They also understand that, by convention, if a number is exactly at the halfway point of two possible answers, the number is rounded up.

Example: *Round 178 to the nearest 10.*



Step 1: The answer is either 170 or 180.

Step 2: The halfway point is 175.

Step 3: 178 is between 175 and 180.

Step 4: Therefore, the rounded number is 180.

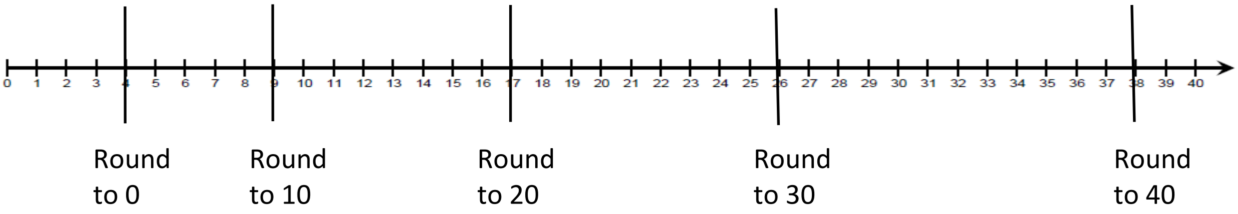
Instructional Strategies:

Prior to implementing rules for rounding students need to have opportunities to investigate and explore place value. A strong understanding of place value is essential for the development of number sense and the subsequent work that involves rounding numbers.

Building on previous understandings of the place value of digits in multi-digit numbers, place value is used to round whole numbers. Dependence on learning rules can be eliminated with strategies such as the use of a number line to determine which multiple of 10 or of 100, a number is nearest (5 or more rounds up, less than 5 rounds down). As students' understanding of place value increases, the strategies for rounding are valuable for estimating, justifying and predicting the reasonableness of solutions in problem-solving.

Continue to use manipulatives like hundreds charts, place-value charts, and number lines.

Below, a number line has been used to show several examples of whole numbers being rounded to the nearest tens place.



Tools / Resources:

See Learning Progressions NBT for detailed information.

Illustrative Mathematics tasks:

- 3.NBT Rounding to 50 or 500
- 3.NBT Rounding to the Nearest Ten and Hundred
- 3.NBT, 4.NBT Rounding to the Nearest 100 and 1000

Also see: "Correcting the Calculator," NCSM, Great Tasks for Mathematics K-5, (2013).

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.NBT.1 to access resources specifically for this standard.

Common Misconceptions: (3.NBT.1 through 3.NBT.3)

The use of terms “round up” and “round down” confuse many students. For example, the number **37** would round to 40 or it “rounds up”. The digit in the tens place is changed from 3 to 4 (rounds up). This misconception is what causes the problem when applied to rounding down. The number **32** should be rounded (down) to 30, but using the logic mentioned for rounding up, some students may look at the digit in the tens place and take it to the previous tens place before 30, resulting in the incorrect value of 20. To remedy this misconception, students need to use a **number line** to visualize the placement of the number and ask questions such as: “*32 comes between which tens? Which ten is it closer to?*”

Developing the **understanding** of the **WHY** behind rounding, what the answer choices are, using place value understanding to round numbers (rather than relying on rounding rhymes e.g. *Find your number, look next door, five or greater add on one more!)* can alleviate much of the misconception and confusion related to rounding.

Domain: Number and Operations in Base Ten (NBT)

● ***Cluster A:*** *Use place value understanding and properties of operations to perform multi-digit arithmetic.*

Standard: 3.NBT.2

Fluently (efficiently, accurately, & flexibly) add and subtract within 1000 using strategies *(e.g. composing/decomposing by like base-10 units, using friendly or benchmark numbers, using related equations, compensation, number line, etc.)* and algorithms (including, but not limited to: traditional, partial-sums, etc.) based on place value, properties of operations, and/or the relationship between addition and subtraction. **(3.NBT.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 3.NBT.1

Explanation and Examples:

This standard refers to fluently, which means with accuracy, efficiency (using a reasonable number of steps and time), and flexibility (using strategies such as the distributive property). The word algorithm refers to a procedure or a series of steps. There are other algorithms other than the standard/traditional algorithm. Third grade students should have experiences beyond the standard/traditional algorithm. In fact, it is argued that students should be introduced to other algorithms (such as partial sums) that are firmly rooted in place value before introducing the traditional algorithm.

Problems should include both vertical and horizontal forms, including opportunities for students to apply the commutative and associative properties. Students explain their thinking and show their work by using strategies and algorithms, and verify that their answer is reasonable.

Example:

There are 178 fourth graders and 225 fifth graders on the playground. What is the total number of students on the playground?

Student 1	Student 2	Student 3	Student 4
$100 + 200 = 300$ $70 + 20 = 90$ $8 + 5 = 13$ $300 + 90 + 13 = 403$ students	I added 2 to 178 to get 180. I added 220 to get 400. I added the 3 left over to get 403.	I know the 75 plus 25 equals 100. I then added 1 hundred from 178 and 2 hundreds from 275. I had a total of 4 hundreds and I had 3 more left to add. So I have 4 hundreds plus 3 more which is 403.	$178 + 225 = ?$ $178 + 200 = 378$ $378 + 20 = 398$ $398 + 5 = 403$

Adding and subtracting fluently refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

Example:

- Mary read 573 pages during her summer reading challenge. She was only required to read 399 pages. How many extra pages did Mary read beyond the challenge requirements?

Students may use several approaches to solve the problem. Examples of these methods are listed below:

- 399 + 1 = 400, 400 + 100 = 500, 500 + 73 = 573, therefore 1+ 100 + 73 = 174 pages (*Adding up strategy*)
- 400 + 100 is 500; 500 + 73 is 573; 100 + 73 is 173 plus 1 (for 399, to 400) is 174 (*Compensation strategy*)
- Take away 73 from 573 to get to 500, take away 100 to get to 400, and take away 1 to get to 399. Then 73 +100 + 1 = 174 (*Subtracting to count down strategy*)
- 399 + 1 is 400, 500 (that's 100 more). 510, 520, 530, 540, 550, 560, 570, (that's 70 more), 571, 572, 573 (that's 3 more) so the total is 1 + 100 + 70 + 3 = 174 (*Adding by friendly numbers strategy*)

Instructional Strategies: (*see cluster 2.NBT.B for strategies that students were introduced to in second grade*)
Strategies used to add and subtract two-digit numbers can now be applied to fluently to add and subtract whole numbers within 1000. These strategies should be discussed so that students can make comparisons and move toward efficient methods.

Addition strategies based on place value for 348 + 537 may include:

- Adding by place value: 300 + 500 = 800 and 40 + 30 = 70 and 8 + 7 = 15. Then 800 + 70 + 15 = **885**.
- Incremental adding (breaking one number into hundreds, tens, and ones): 537 + 100 = 637, 637 + 100 = 737, 737 + 100 = 837. Then 837 + 10 = 847, 847 + 10 = 857, 857 + 10 = 867, 867 + 10 = 877. Then 877 + 8 = **885**.
- Compensation (making a friendly number): Take 2 from 537 and move those 2 to the 348 to make 350 + 535. Much easier to add these to get **885**. Definition of compensation method would be helpful.

Subtraction strategies based on place value for 81 - 37 may include:

- Adding up (from smaller number to larger number): 37 + 3 = 40, 40 + 40 = 80, 80 + 1 = 81, and 3 + 40 + 1 = **44**.
- Incremental subtracting: 81 -10 = 71, 71 – 10 = 61, 61 – 10 = 51, 51 – 7 = **44.**
- Subtracting by place value: 81 – 30 = 51, 51 – 7 = **44.**

Number sense and computational understanding is built on a firm understanding of **place value**.

Resources/Tools:

Illustrative Mathematics tasks:

- 3.OA, MD, NBT Classroom Supplies

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.NBT.2 to access resources specifically for this standard.

Thinking Blocks on Math Playground allows students several ways to model problems.

Common Misconceptions: See 3.NBT.1

Students may think that the 4 in 46 represents 4, not 40 or 4 tens. Students need many experiences representing two- and three-digit numbers with manipulatives that group (base ten blocks) and those that do NOT group, such as counters, etc.

Domain: Number and Operations in Base Ten (NBT)

● ***Cluster A:*** *Use place value understanding and properties of operations to perform multi-digit arithmetic.*

Standard: 3.NBT.3

Multiply one-digit whole numbers by multiples of 10 in the range 10 to 90 (*e.g.* $9 \cdot 80$, $5 \cdot 60$) using strategies based on place value and properties of operations. **(3.NBT.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 3.NBT.1

Explanation and Examples:

This standard extends students' work in multiplication by having them apply their understanding of place value. This standard expects students to go beyond tricks that hinder understanding such as "just adding zeroes" and explain and reason about their products. *For example, in the problem 50 x 4, students should think of this as 4 groups of 5 tens or 20 tens. Twenty tens equals 200.*

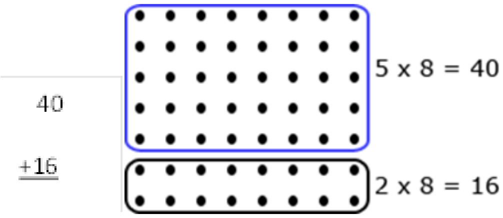
Students use base ten blocks, diagrams, or hundreds charts to multiply one-digit numbers by multiples of 10 from 10-90. They apply their understanding of multiplication and the meaning of the multiples of 10. For example, 30 is 3 tens and 70 is 7 tens. They can interpret 2 x 40 as 2 groups of 4 tens or 8 groups of ten. They understand that 5 x 60 is 5 groups of 6 tens or 30 tens and know that 30 tens is 300. After developing this understanding they begin to recognize the patterns in multiplying by multiples of 10.

> ******* As the teacher, one of the KEY understandings of multiplicative reasoning you want to develop in your students is that multiplication extends beyond repeated addition. If students do not move from additive to multiplicative thinking then their development in understanding higher mathematics will be compromised. Students should understand that there is a *multiplicative unit* and a *scaling factor* in the following mathematical expression – **3 x 50**. **50** or **5 tens** is the multiplicative unit (*multiplicand*) and **3** is the scaling factor (*multiplier*) for that multiplicative unit. Essentially the expression is telling you that there are "3 groups of 50/5 tens" or "3 copies of 50/5 tens". This seems it is being repeated from previous.

Instructional Strategies:

Understanding what each number in a multiplication expression represents is important. Multiplication problems need to be modeled with pictures, diagrams or concrete materials to help students understand what the factors and products represent. The effect of multiplying numbers needs to be examined and understood.

The use of area models is important in understanding the properties of operations of multiplication and the relationship of the factors and its product. Composing and decomposing area models is useful in the development and understanding of the distributive property in multiplication.



Resources/Tools:

See EngageNY Modules

Illustrative Mathematics tasks:

- 3.NBT How Many Colored Pencils?

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.NBT.3 to access resources specifically for this standard.

Thinking Blocks on Math Playground allows students several ways to model problems.

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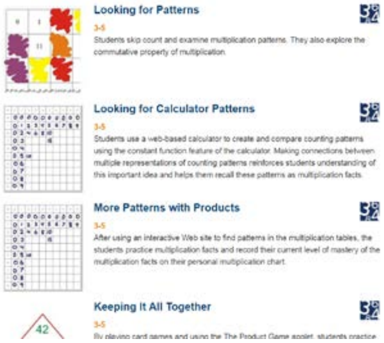
- Multiplication: It's in the Cards
  (view of some of the lessons)



Looking for Patterns

3-5

Students skip count and examine multiplication patterns. They also explore the commutative property of multiplication.

Looking for Calculator Patterns

3-5

Students use a web-based calculator to create and compare counting patterns using the constant function feature of the calculator. Making connections between multiple representations of counting patterns reinforces students understanding of this important idea and helps them recall these patterns as multiplication facts.

More Patterns with Products

3-5

After using an interactive Web site to find patterns in the multiplication tables, the students practice multiplication facts and record their current level of mastery of the multiplication facts on their personal multiplication chart.

Keeping It All Together

3-5

By playing card games and using the The Product Game applet, students practice

Common Misconceptions: See 3.NBT.1

Domain: Number and Operations—Fractions (NF)

▶ ***Cluster A:*** *<u>Develop understanding of fractions as numbers.</u>*
Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

Standard: 3.NF.1

Understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction $\frac{a}{b}$ as the quantity formed by *a* parts of size $\frac{1}{b}$. **(3.NF.1)**

Suggested Standards for Mathematical Practice (MP):

✓ MP.1 Make sense of problems and persevere in solving them.
✓ MP.4 Model with mathematics.
✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected to:

- Partitioning traditional shapes into equal parts (Grade 1.G.3) & Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths (2.G.3).

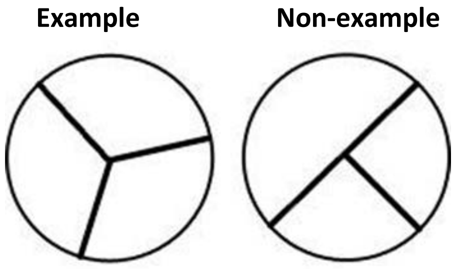
Explanation and Examples:

Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.
This standard refers to the sharing of a whole being partitioned or split. Fraction models in third grade include area models (circles, rectangles, squares, etc.) and linear models (linear, measurement). **Set models** (parts of a group) are not expected to be mastered in Third Grade.

In 3.NF.1 students should focus on the concept that a fraction is made up (composed) of many pieces of a **unit fraction**, which has a numerator of 1. For example, the fraction $\frac{3}{5}$ is composed of 3 pieces that each have a size of $\frac{1}{5}$.

Some important concepts related to developing understanding of fractions include:

- Understand fractional parts must be equal-sized

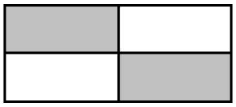

Example **Non-example**

- The number of equal parts tells how many make a whole.
- As the number of equal pieces in the whole increases, the size of the fractional pieces decreases.
- The size of the fractional part is relative to the whole.

To develop understanding of fair shares, students first participate in situations where the number of equal parts is greater than the number of children and then progress into situations where the number of equal parts is less than the number of children.

Examples of Area or Region Models:

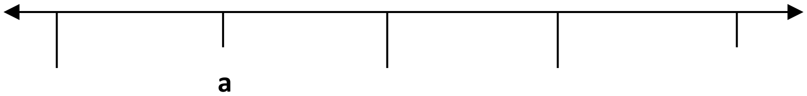
- Four children share a pan of brownies so that each child receives a fair share. How much of the pan of brownies will each child receive?
- Six children share two pans of brownies so that each child receives a fair share. What portion will each child receive?
- What fraction of the rectangle is shaded? How might you shade the rectangle in another way but end up with the same fraction shaded?



Solution: $\frac{2}{4}$ or $\frac{1}{2}$

Example of a Linear Model:

What fraction does the letter **a** represent on this number line? (Linear Model) Explain your thinking.



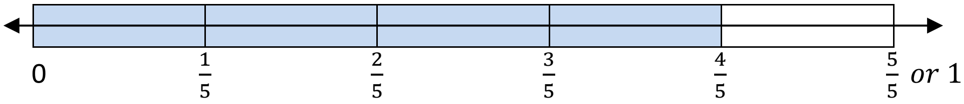
Instructional Strategies: (3.NF.1 through 3.NF.3)

Understanding fractions is an essential element if students are to be successful in higher mathematics. 3rd grade lays the foundation for this understanding so the **use of concrete manipulatives, visuals, diagrams, and language cannot be overemphasized**.

This is the initial experience students will have with fractions and instruction is best implemented over an extended period of time. Students need many opportunities to discuss fractional parts using concrete models to develop familiarity and understanding of fractions.

Understanding that a fraction is a quantity formed by part of a whole is essential to number sense with fractions. Fractional parts are the building blocks for all fraction concepts. Students need to relate dividing a shape into equal parts and representing this relationship on a number line, where the equal parts are between two whole numbers.

Help students plot fractions on a number line, by using the meaning of the fraction. For example, to plot $\frac{4}{5}$ on a number line, there are 5 equal parts with 4 copies of the 5 equal parts. 5 equal parts make the **whole**. The **unit fraction** is $\frac{1}{5}$.



4 copies of the 5 equal parts represent the fractional amount shown on this number line.

Knowing the whole and the unit fraction is critical when understanding and working with fractions.

As students counted with whole numbers, they should also count with fractions. Counting equal-sized parts helps students determine the number of parts it takes to make a whole and recognize fractions that are equivalent to whole numbers. Make sure you count beyond the whole number 1. Too frequently students believe fractions only exist between 0 and 1. They must build the understanding that fractions definitely more beyond 1.

Tools / Resources

Illustrative Mathematics tasks:

- 3.NF Naming the Whole for a Fraction
- 3.MD, 3.G, 3.NF Halves, thirds, and sixths

See: "Fraction Reactions," NCSM, Great Tasks for Mathematics K-5, (2013).

For detailed explanations and examples see the Number and Operations - Fractions learning progression.

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.NF.1 to access resources specifically for this standard.

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- Access this link to view many quality lessons and interactives focusing on fractions.

Common Misconceptions:

The idea that the smaller the denominator, the smaller the piece, or the larger the denominator, the larger the piece, is based on the thinking and reasoning students used with working with whole numbers (the smaller a number, the less it is, or the larger a number, the more it is). The use of different models, such as fraction bars and number lines, allows students to compare unit fractions to reason about their sizes and correct this misconception.

Students think all shapes can be divided the same way. Present shapes other than circles, squares or rectangles to prevent students from over generalizing that all shapes can be divided the same way. For example, have students fold a triangle into eighths. Provide oral directions for folding the triangle:

1. Fold the triangle into half by folding the left vertex (at the base of the triangle) over to meet the right vertex.
2. Fold in this manner two more times.
3. Have students label each eighth using fractional notation. Then, have students count the fractional parts in the triangle (one-eighth, two-eighths, three-eighths, and so on).

Students frequently will count "tick marks" on number lines (linear/length model) rather than the distance or region partitioned. Fractional pieces is the space between the tick marks, not the tick marks themselves.

Domain: Number and Operations—Fractions (NF)

▶ ***Cluster A:*** *Develop understanding of fractions as numbers.*

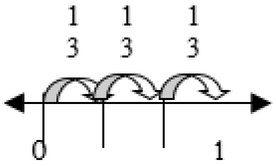
Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

Standard: 3.NF.2

Understand a fraction as a number on the number line; represent fractions on a number line diagram.

3.NF.2a. Represent a fraction $\frac{1}{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line. **(3.NF.2a)**

Ex:



3.NF.2b. Represent a fraction $\frac{a}{b}$ on a number line diagram by marking off a lengths $\frac{1}{b}$ from 0. Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the number $\frac{a}{b}$ on the number line (a is the countable units of $\frac{1}{b}$ that determines the place on the number line). **(3.NF.2b)**

Suggested Standards for Mathematical Practice (MP):

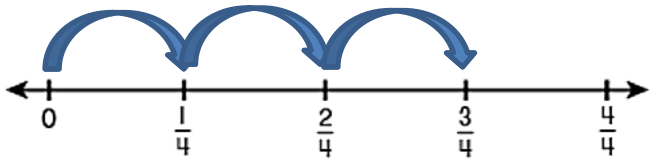
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections: See 3.NF.1

Explanation and Examples:

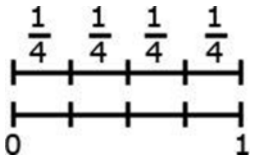
Third grade is the first time students will work with a number line for numbers that are between whole numbers (e.g., that $\frac{1}{2}$ is between 0 and 1).

In the number line diagram below, the space between 0 and 1 is divided (partitioned) into 4 equal regions. The distance from 0 to the first segment is *1 of the 4* segments from 0 to 1 or $\frac{1}{4}$ (**3.NF.2a**). Similarly, the distance from 0 to the third segment is 3 segments that are each one-fourth long. Therefore, the distance of 3 segments from 0 is the fraction $\frac{3}{4}$ (**3.NF.2b**).

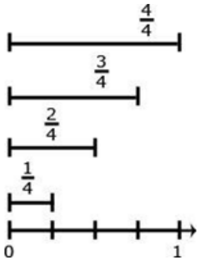


Students transfer their understanding of parts of a whole to partition a number line into equal parts. There are two new concepts addressed in this standard which students should have time to develop.

1. On a number line from 0 to 1, students can partition (divide) it into equal parts and recognize that each segmented part represents the same length.



2. Students label each fractional part based on how far it is from zero to the endpoint.



Instructional Strategies: See 3.NF.1

Tools / Resources:

See: "Fraction Reactions," NCSM, Great Tasks for Mathematics K-5, (2013).

Illustrative Mathematics tasks:

- 3.NF Locating Fractions Less than One on the Number Line
- 3.NF Closest to 1/2
- 3.NF Locating Fractions Greater than One on the Number Line
- 3.NF Find 1
- 3.NF Find 2/3
- 3.NF Which is Closer to 1?
- 3.NF Find 1/4 Starting from 1, Assessment Version
- 3.NF Find 7/4 starting from 1, Assessment Variation
- 3.NF Find 1 Starting from 5/3, Assessment Variation

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.NF.2 to access resources specifically for this standard.

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- Access this link to view many quality lessons and interactives focusing on fractions.

Common Misconceptions: See 3.NF.1

Domain: Number and Operations—Fractions (NF)

▶ ***Cluster A:*** *Develop understanding of fractions as numbers.*

Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

Standard: 3.NF.3

Explain equivalence of fractions, and compare fractions by reasoning about their size (it is a mathematical convention that when comparing fractions, the whole is the same size).

- 3.NF.3a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line. **(3.NF.3a)**
- 3.NF.3b. Recognize and generate simple equivalent fractions, *(e.g.* $\frac{1}{2}=\frac{2}{4}, \frac{4}{6}=\frac{2}{3}$*.)* Explain why the fractions are equivalent, *e.g. by using a visual fraction model*. **(3.NF.3b)**
- 3.NF.3c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form* $3=\frac{3}{1}$*; recognize that* $\frac{6}{1}=6$*; locate* $\frac{4}{4}$ *and 1 at the same point of a number line diagram.* **(3.NF.3c)**
- 3.NF.3d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the relational symbols >, <, =, or ≠, and justify the conclusions, *(e.g. by using a visual fraction model.)* **(3.NF.3d)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 3.NF.1

Explanation and Examples:

Equivalence is a core concept in mathematics. Students must not be shortchanged in the amount of time needed to work to develop a firm understanding of this concept. Often students who claim to understand what the equal sign means in an equation (6 x 4 = 24), will be confused when the equation is given as 24 = 6 x 4. This confusion can get further muddled if time is not given when working with equivalence in fractions.

As adults, we know that when comparing fractional amounts, the **whole must be the same**. Often this assumption is continued in textbooks. It will be stated that $\frac{1}{2}$ is equal to $\frac{2}{4}$, but this only true when they are referring to the same whole. This assumption cannot be made with children. We need to be explicit when comparing fractions.

An important concept when comparing fractions is to look at the size of the parts and the number of the parts. For example, $\frac{1}{8}$ is smaller than $\frac{1}{2}$ because when 1 whole is cut into 8 pieces, the pieces are much smaller than when the same

1 whole is cut into 2 pieces. (Students can SEE this by modeling----folding the same size paper in half, in half again, and so on.)

3.NF.3a and **3.NF.3b** expect students to use visual fraction models (such as area models), number lines, and reasoning about their size to explore the idea of equivalent fractions. Students at this level should explore equivalent fractions **using models**, rather than using algorithms.

When using reasoning to compare fractions, students can think of **benchmarks**. For example, I can compare $\frac{5}{6}$ and $\frac{3}{4}$ by thinking about their distance from 1 on a number line. $\frac{5}{6}$ is only $\frac{1}{6}$ away from 1, but $\frac{3}{4}$ is $\frac{1}{4}$ away from 1. $\frac{1}{4}$ is a larger portion than $\frac{1}{6}$ so $\frac{3}{4}$ is a greater distance away from 1. This means that $\frac{5}{6}$ is greater than $\frac{3}{4}$.

In this standard, students should also reason that comparisons are only valid if the wholes are identical. For example, $\frac{1}{2}$ of a large pizza is a different amount than $\frac{1}{2}$ of a small pizza. Students should be given opportunities to discuss and reason about which $\frac{1}{2}$ is larger.

An important concept when comparing fractions is to look at the size of the parts and the number of the parts. For example, $\frac{1}{8}$ is smaller than $\frac{1}{2}$ because when 1 whole is cut into 8 pieces, the pieces are much smaller than when 1 whole is cut into 2 pieces.

Students recognize when examining fractions with common denominators, the wholes have been divided into the same number of equal parts. So the fraction with the larger numerator has the larger number of equal parts.

$$\mathbf{\frac{2}{6} < \frac{5}{6}}$$

To compare fractions that have the same numerator but different denominators, students understand that each fraction has the same number of equal parts but the size of the parts are different. They can infer that the same number of smaller pieces is less than the same number of bigger pieces.

$$\mathbf{\frac{3}{8} < \frac{3}{4}}$$

This standard also includes writing whole numbers as fractions. The concept relates to fractions as division problems, where the fraction $\frac{3}{1}$ is 3 wholes divided into one group. This standard is the building block for later work where students divide a set of objects into a specific number of groups. Students must understand the meaning of $\frac{a}{1}$.

Instructional Strategies: See 3.NF.1

Tools/Resources:

Illustrative Mathematics tasks:

- 3.NF Ordering Fractions
- 3.NF Comparing Fractions
- 3.NF Snow Day
- 3.NF Jon and Charlie's Run
- 3.MD, 3.G, 3.NF Halves, thirds, and sixths
- 3.NF Comparing Fractions with a Different Whole
- 3.NF Comparing Fractions with the Same Denominator, Assessment Variation
- 3.NF Comparing Fractions with the Same Numerators, Assessment Variation
- 3.NF Fraction Comparisons With Pictures, Assessment Variation

Georgia Department of Education:

- "Making a Cake"

See: Grade 3-5 Number and Operations Fractions Learning Progressions for detailed information:

Visit K-5 Math Teaching Resources click on **Number**, then on **3rd Grade**. Scroll down to 3.NF.3 to access resources specifically for this standard.

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- Access this link to view many quality lessons and interactives focusing on fractions.
- Fraction Game.

Common Misconceptions: See 3.NF.1

Misunderstanding the meaning of the equal size is the most common misconception for students. Make sure you spend enough time working with models and visuals so students can build a firm foundation of equality and then build on the understanding of inequality.

Another misconception is not understanding that when comparing fractions the wholes have to be the same. $\frac{1}{2}$ of a large pizza is very different than $\frac{1}{2}$ of a small pizza. Discussing this idea with your students is a critical foundational piece of learning.

Domain: Measurement and Data (MD)

▶ ***Cluster A:*** *Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.*

Standard: 3.MD.1

Tell and write time to the nearest minute using a.m. and p.m. and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, *(e.g. by representing the problem on a number line diagram.)* (See Table 1) **(3.MD.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: (3.MD.1)

This standard is related to:

- Work with time and money in Grade 2 (2.MD.7)

Explanation and Examples:

This standard expects students to solve elapsed time, including word problems. Students can use clock models or number lines to solve. On the number line, students should be given the opportunities to determine the intervals and size of jumps on their number line. Students could use pre-marked number lines (intervals every 5 or 15 minutes) or open number lines (intervals determined by students).

Students in second grade learned to tell time to the nearest five minutes. In third grade, they extend telling time and measure elapsed time both in and out of context using clocks and number lines. They are to also distinguish between a.m. and p.m.

Instructional Strategies: (3.MD.1)

A clock is the common instrument for measuring time. Learning to tell time has much to do with learning to read a dial-type instrument rather than with time measurement. Students should develop the understanding that an analog clock is essentially a number line that has been formed into a circle. The grouping (or bundling) of time is different in that hours are grouped by 12s or 24s and minutes are grouped by 60s.

Students have experience in telling and writing time from analog and digital clocks to the hour and half hour in Grade 1 and to the nearest five minutes in Grade 2. Now students will tell and write time to the nearest minute (distinguishing between a.m. and p.m.) and measure time intervals in minutes.

Providing geared analog clocks allows students to understand the movement of the minute hand. If students are struggling with telling time, try the "One-handed Clock" lesson provided in Dr. John Van de Walle's book, Teaching Student-Centered Mathematics PreK – 2. The hour hand gives the most information about the time. To give students a better understanding of this you will need to buy two inexpensive clocks. Place both clocks in an area so all students can see them but are easy for you to access. Make sure both clocks are set to the same correct time and then remove the minute hand from one of the clocks. The clock with both hands should then be covered so that students will see just the one-handed clock. At various times during the day, draw your students' attention to the one-handed clock and ask them to tell you the time. Then remove the cover from the two-handed clock to verify the time. Students will begin to see that the hour hand gives them an idea of how many minutes past the hour it is based on how far it is between two numbers.

Students need experience representing time from a digital clock to an analog clock and vice versa.

Provide word problems involving addition and subtraction of time intervals in minutes. Have students represent the problem on a number line.

Resources/Tools:

For detailed information see Measurement Learning Progression

See EngageNY Modules

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **3rd Grade**. Scroll down to 3.MD.1 to access resources specifically for this standard.

Illustrative Mathematics tasks:

- Dajuana's Homework

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- Elapsed Time

Thinking Blocks on Math Playground allows students several ways to model problems.

Number line tool for elapsed time:
Elapsed Time: How to Solve Elapsed Time on a Number Line

Common Misconceptions:

Avoid the use of paper plate clocks. Students need to see the actual relationship between the hour and the minute hand. This is not adequately represented on student-made clocks since there are not gears to move the hands so that they are in concert with each other. When students represent the time, they frequently put the hour hand on the whole number whether it is on the hour or half-past the hour. Using geared clocks will avoid this misconception. (See the **One-Handed Clock** lesson referred to in the *Instructional Strategies* section above.)

Domain: Measurement and Data (MD)

▶ *Cluster A:* *Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.*

Standard: 3.MD.2

Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l) (Excludes cubed units such as $cm^3$and finding the geometric volume of a container). **(3.MD.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections:

This standard is foundational to:

- Measurement and Data standard in 4th grade (4.MD.2)
- Measurement and Data standards in 5th grade (5.MD.3, 5.MD.4, 5.MD.5)

Explanation and Examples:

This standard asks for students to reason about the units of mass and volume. Students need multiple opportunities weighing classroom objects and filling containers to help them develop a basic understanding of the size and weight of a liter, a gram, and a kilogram. Milliliters may also be used to show amounts that are less than a liter. Word problems should only be one-step and include the same units.

Example:

Students identify 5 things that have a mass of about one gram. They record their findings with words and pictures~~.~~ (Students can repeat this for 5 grams and 10 grams~~.~~). This activity helps develop gram benchmarks and assists in estimation activities. One large paperclip weighs about one gram. A box of large paperclips (100 clips) has a mass of about 100 grams so 10 boxes would have a mass of one kilogram.

Foundational understandings to help with measure concepts:

- Understand that larger units can be subdivided into equivalent units (partition).
- Understand that the same unit can be repeated to determine the measure (iteration).
- Understand the relationship between the size of a unit and the number of units needed (compensatory principle).

Instructional Strategies:

Students need multiple opportunities "massing" classroom objects and filling containers to help them develop a basic understanding of the size and mass of a liter, a gram, and a kilogram. Milliliters may also be used to show amounts that are less than a liter.

Provide opportunities for students to use appropriate tools to measure and estimate liquid volumes in liters only and masses of objects in grams and kilograms. Students need practice in reading the scales on measuring tools since the markings may not always be in intervals of one. The scales may be marked in intervals of two, five or ten.

Allow students to hold gram and kilogram weights in their hands to use as a benchmark for estimation. Use water colored with food coloring so that the water can be seen in a beaker.

Students should **estimate** volumes and masses before finding the exact measures. Show students a group of objects (all are the same object such as a grouping of small water bottles or same size jewelry boxes. Then, indicate one of the objects and tell the students its weight. Fill a box with more of the same objects and ask students to estimate the weight of them.

Use similar strategies with liquid measures. Be sure that students have opportunities to pour liquids into different size containers to see how much liquid will be in certain whole liters. Show students containers and ask, “How many liters do you think will fill this container?”

If estimating several containers, students should make an estimate, then complete the measurement. They can then continue the process of estimating and then measuring, rather than all estimates and then all measures. It is important to provide feedback to students on their estimates by using measurement as a way of gaining feedback on estimates.

Tools/Resources:

See K-5 Measurement Learning Progressions for detailed information.

Illustrative Math site task:

- 3.MD How Heavy?

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **3rd Grade**. Scroll down to 3.MD.2 to access resources specifically for this standard.

Common Misconceptions:

Students often focus on size to determine estimates of mass. They can be confused by a big fluffy object and a tiny dense object. Because students cannot tell actual mass until they have handled an object, it is important that teachers do not ask students to estimate the mass of objects until they have had the opportunity to lift the objects and then make an estimate of the mass.

Students may read the mark on a scale that is below a designated number on the scale as if it was the next number. For example, a mark that is one mark below 80 grams may be read as 81 grams. Students realize it is one away from 80, but do not think of it as 79 grams.

Domain: Measurement and Data (MD)

▶ ***Cluster A:*** *Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.*

Standard: 3.MD.3

Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, *(e.g. by using drawings (such as a beaker with a measurement scale) to represent the problem.)* (Excludes multiplicative comparison problems) (See Table 1 and Table 2). **(3.MD.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections:

This standard is related to:

- Problem solving standards in 2nd grade (2.OA.1) & in 3rd grade (3.OA.8).

This standard is foundational to:

- Measurement and Data standard in 4th grade (4.MD.2)
- Measurement and Data standards in 5th grade (5.MD.3, 5.MD.4, 5.MD.5)

Explanation and Examples:

This standard excludes multiplicative comparison problems (problems involving notions of “times as much”). These types of problems will be learned in 4th grade.

This standard is building on the work from 1st and 2nd grade in solving problems in context (word problems) – 1.OA.1 & 2.OA.1 – and focusing on masses and liquid volumes (capacities) but not excluding other areas of measurement. With the focus on measurement, the role of the unit must be emphasized. Students in third grade are not required to convert so units must be the same.

Examples:

If I need 48 cups of lemonade to bring to class during field day and I have containers that hold 8 cups each, how many containers will I need to bring with me that day?

You have 21 inches of string. Your best friend cut it into 3 equal pieces. How long is each piece?

A dump truck brought a load of rocks to school. The principal put 284 pounds of the rock into the front garden area. The custodian put 545 pounds of rock in the planters around the driveway. The science teacher said there is 125 pounds left to use. How much did the dump truck bring to school?

Instructional Strategies:

Students need to have multiple opportunities to solve problems in context and discuss the units involved. Estimation should also be used so students have an idea of what would be a reasonable answer. Don't expect an exact answer. Ask students to think about and provide a range of where the answer would be. Student discussions are critical in establishing an environment that supports estimation and backing up their thinking with evidence.

Tools/Resources:

Thinking Blocks on Math Playground allows students several ways to model problems.

Greg Tang's Word Problem Generator

See K-5 Measurement Learning Progressions for detailed information.

Domain: Measurement and Data (MD)

◆ ***Cluster B:*** ***<u>Represent and interpret data.</u>***

Standard: 3.MD.4

Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step “how many more” and “how many less” problems using information presented in scaled bar graphs. (See Table 1). *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.* **(3.MD.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: (3.MD.4 & 3.MD.5)

This cluster is connected to:

- Represent and solve problems involving multiplication and division. (Grade 3.OA.1 through 3.OA.4)
- Multiply and divide within 100. (Grade 3.OA.7)
- Solve problems involving the four operations, and identify and explain patterns in arithmetic. (Grade 3.OA.8 & 3.OA.9)
- Represent and interpret data. (Grade 2.MD.10 and 2.MD.11)

Explanation and Examples:

Students should have opportunities reading and solving problems using scaled graphs before being asked to draw one. Graphs on the next page all use five as the scale interval, but students should experience different intervals to further develop their understanding of scale graphs and number facts.

While exploring data concepts, students should 1) **Pose** a question, 2) **Collect** data, 3) **Analyze** data, and 4) **Interpret** data**.** Students should be graphing data that is relevant to their lives.

Example:

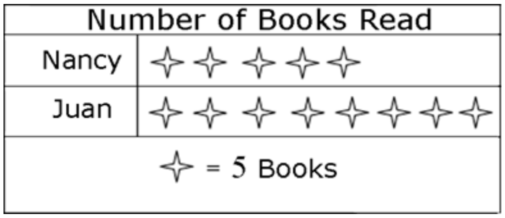
The teacher can pose a question that will lead students to want to investigate and collect information: Do all students in our school like the same type~~s~~ of books? What type~~s~~ of books should we recommend that the librarian (or principal or PTA) order for the library?

Students should come up with questions. What is the typical type of book read in our school?

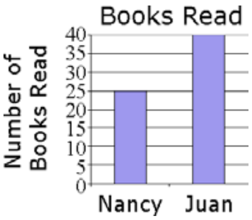
Then students need to collect and organize data. They can create student surveys that can be delivered in paper form or in electronic form.

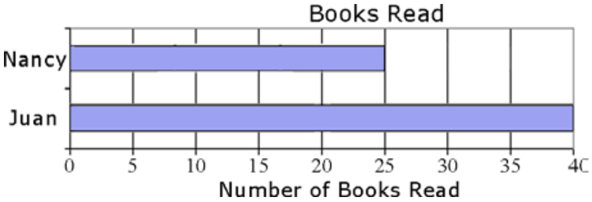
Pictographs: Scaled pictographs include symbols that represent multiple units. Below is an example of a pictograph with symbols that represent multiple units. Graphs should include a title, scale, categories, category label, and data. Students need to use both horizontal and vertical pictographs.

Example of Scaled Pictograph:



- Bar Graphs (these examples are scaled bar graphs): Students should use both horizontal and vertical bar graphs. Bar graphs include a title, scale, scale label, categories, category label, and data.





Analyzing and Interpreting data could include:

- How many more nonfiction books where read than fantasy books?
- Did more people read biography and mystery books or fiction and fantasy books?
- About how many books in all genres were read?
- Using the data from the graphs, what type of book was read more often than a mystery but less often than a fairytale?
- What interval was used for this scale?
- What can we say about types of books read? What is a typical type of book read?
- If you were to purchase a book for the school library which would be the best genre?

Instructional Strategies: (3.MD.4 and 3.MD.5)

Representation of a data set is extended from picture graphs and bar graphs with single-unit scales to scaled picture graphs and scaled bar graphs. Intervals for the graphs should relate to multiplication and division with 100 (product is 100 or less and numbers used in division are 100 or less).

Students are to draw picture graphs in which a symbol or picture represents more than one object. Bar graphs are drawn with intervals greater than one. Ask questions that require students to compare quantities and use mathematical concepts and skills. Use symbols on picture graphs that student can easily represent half of, or know how many half of the symbol represents.

In picture graphs, you could use values for the icons in which students need practice with their multiplication facts. For example, ⍰ represents 7 people. If there are three ⍰, students will need to use known facts to determine that the three icons represents 21 people. The intervals on the vertical scale in bar graphs should not exceed 100.

*** Note: If a scale is used that is not easily divided into half then only full pictures should be used for that graph. For example, the situation above had each picture represent 7 people, so this scale would not be used with half pictures since you cannot have half of a person.

Resources/Tools

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- "Bar Grapher", This is a NCTM site that contains a bar graph tool to create bar graphs.
- Picture This
- "It's All About Multiplication-Exploring Equal Sets" – Students listen to the counting story, *What Comes in 2's, 3's, & 4's*, and then use counters to set up multiple sets of equal size. They fill in a table listing the number of sets, the number of objects in each set, and the total number in all. They study the table to find examples of the order (commutative) property. Finally, they apply the equal sets model of multiplication by creating pictographs in which each icon represents several data points.
- "What's in a Name? – Creating Pictographs" – This is a series of lessons in which students use data tools, one of which is pictographs, and answer questions about the data set.

Georgia Department of Education:

- "Barnyard Legs" - Students solve multiplication problems using different strategies based on Amanda Bean's Amazing Dream, A Mathematical Story by Cindy Neuschwander or a similar book about multiplication.
- "Guess Who's Coming to Dinner" Students are to arrange 18 people at 6 different card tables. Each table must be full and there must be an adult at each table. Students will use perimeter to find the solution.

Illustrative Mathematics tasks:

- 3.OA, MD, NBT Classroom Supplies

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **3rd Grade**. Scroll down to 3.MD.3 to access resources specifically for this standard.

Common Misconceptions:

Although intervals on a bar graph are not in single units, students count each square as one. To avoid this error, have students include tick marks between each interval. Students should begin each scale with 0. They should think of skip-counting and then connect to multiplication when determining the value of a bar since the scale is not in single units.

Domain: Measurement and Data (MD)

◆ ***Cluster B:*** *Represent and interpret data.*

Standard: 3.MD.5

Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters. **(3.MD.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See 3.MD.4

Explanation and Examples:

Students in second grade measured length in whole units using both metric and U.S. customary systems. It's important to review with students how to read and use a standard ruler including details about halves and quarter marks on the ruler. Students should connect their understanding of fractions to measuring to one-half and one-quarter inch. Third graders need many opportunities measuring the length of various objects in their environment.

Students are to measure lengths using rulers marked with halves and fourths of an inch and record the data on a line plot. The horizontal scale of the line plot is marked off in whole numbers, halves or fourths. Students can create rulers with appropriate markings and use the ruler to create the line plots.

This standard provides a context for students to work with fractions by measuring objects to a quarter of an inch.

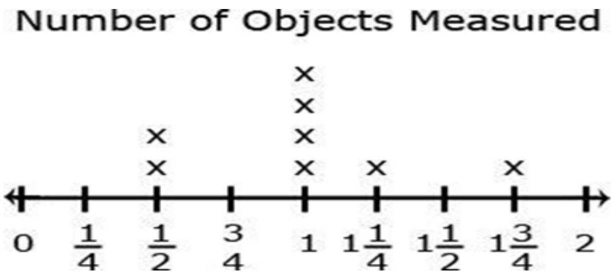
Example:

Measure objects in your desk to the nearest $\frac{1}{2}$ or $\frac{1}{4}$ of an inch, display data collected on a line plot. How many objects measured $\frac{1}{4}$? $\frac{1}{2}$? etc.

Some important ideas related to measuring with a ruler are:

- The starting point where the ruler is placed to begin measuring.
- Measuring is approximate. Items that students measure will not always measure **exactly** $\frac{1}{4}$, $\frac{1}{2}$ or one whole inch. Students will need to decide on an appropriate length estimate.
- Making paper rulers and folding to find the half and quarter marks will help students develop a stronger understanding of measuring length

Students generate data by measuring and creating a line plot to display their findings. An example of a line plot is shown below:



Instructional Strategies: See 3.MD.4

Tools/Resources:

See EngageNY Modules

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **3rd Grade**. Scroll down to 3.MD.4 to access resources specifically for this standard.

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- Inch by Inch

Domain: Measurement and Data (MD)

▶ ***Cluster C:*** *Geometric measurement: understand concepts of area and relate area to multiplication and to addition.*

Standard: 3.MD.6

Recognize area as an attribute of plane figures and understand concepts of area measurement.

- 3.MD.6a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area (does not require standard square units). **(3.MD.5a)**
- 3.MD.6b. A plane figure which can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units (does not require standard square units). **(3.MD.5b)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: (3.MD.6 through 3.MD.8)

This cluster is connected to:

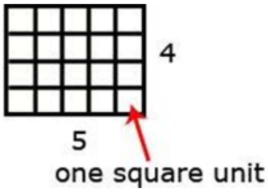
- Third Grade standard for multiplication using arrays and the area model (3.OA.3).
- Fluently multiply and divide within 100 (3.OA.7).
- Distributive property (3.OA.5).

Explanation and Examples: (3.MD.6 through 3.MD.8)

These standards expect students to explore the concept of covering a region with "unit squares," which could include square tiles or shading on grid or graph paper.

Students develop an understanding of using square units in order to measure area:

- Using different sized square units to explore area but knowing that the same unit must be used when measuring (color tiles, squares cut from construction paper, patty paper squares, etc.).
- Filling in an area with the same sized square units and counting the number of square units.
- An interactive whiteboard would allow students to see that square units can be used to cover a plane figure.



Instructional Strategies: (3.MD.6 through 3.MD.8)
Students can cover rectangular shapes with tiles and count the number of units (tiles) to begin developing the idea that area is a measure of covering. Area describes the size of the inside space of an object that is two-dimensional. The formulas should not be introduced before students explore and uncover the meaning of area for themselves.

The area of a rectangle can be determined by having students lay out unit squares and count how many square units it takes to completely cover the rectangle completely without overlaps or gaps.

Students need to develop the meaning for computing the area of a rectangle. A connection needs to be made between the *number of squares* it takes to cover the rectangle and the *dimensions* of the rectangle. Ask questions such as:

- What does the length of a rectangle describe about the squares covering it?
- What does the width of a rectangle describe about the squares covering it?

Tools/Resources:

Illustrative Math Task:

- 3.MD The Square Counting Shortcut

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **3rd Grade**. Scroll down to 3.MD.5 to access resources specifically for this standard.

Common Misconceptions:

Students may confuse perimeter and area when they measure the sides of a rectangle and then multiply. They think the attribute they find is length, which is perimeter. Pose problems situations that require students to explain whether they are to find the perimeter or area.

Domain: Measurement and Data (MD)

▶ ***Cluster C:*** *Geometric measurement: understand concepts of area and relate area to multiplication and to addition.*

Standard: 3.MD.7

Measure areas by counting unit squares (square cm, square m, square in, square ft, and non-standard square units). **(3.MD.6)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See 3.MD.6

Explanation and Examples:

Students should be counting the square units to find the area could be done in metric, customary, or non-standard square units. Using different sized graph paper, students can explore the areas measured in square centimeters and square inches.

An interactive whiteboard may also be used to display and count the unit squares (area) of a figure.

Tools/Resources:

Illustrative Math Tasks:

- 3.MD, 3.G, 3.NF Halves, thirds, and sixths
- 3.MD Finding the Area of Polygons

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **3rd Grade**. Scroll down to 3.MD.6 to access resources specifically for this standard.

Common Misconceptions: See 3.MD.6

Domain: Measurement and Data (MD)

▶ ***Cluster C:*** *Geometric measurement: understand concepts of area and relate area to multiplication and to addition.*

Standard: 3.MD.8

Relate area to the operations of multiplication and addition
(Measurement and Data (measurement part) Progression K–5 Pg. 16).

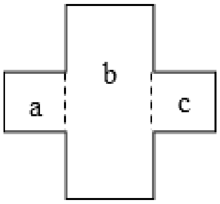
3.MD.8a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths. **(3.MD.7a)**

3.MD.8b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning. **(3.MD.7b)**

3.MD.8c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \cdot b$ and $a \cdot c$. Use area models to represent the distributive property in mathematical reasoning (Supports 3.OA.5). **(3.MD.7c)**

3.MD.8d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems. **(3.MD.7d)**

Example:



Students can find the total area of the shape by finding the areas of a, b, and c and adding them together.

Suggested Standards for Mathematical Practice (MP):

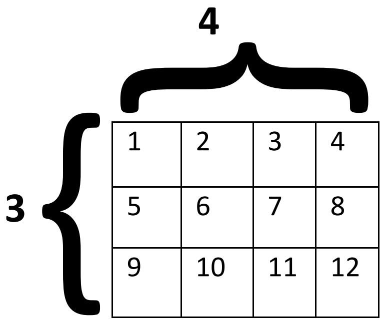
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See 3.MD.6

Explanation and Examples:

Students should tile a rectangle then multiply the side lengths to show that they come up with the same number of squares.

To find the area one could count the squares (as indicated by the numbers on each square) or multiply 3 by 4 to get a total number of squares of 12.

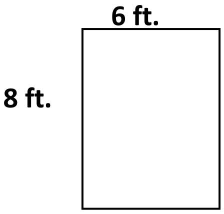


1	2	3	4
5	6	7	8
9	10	11	12

Students should solve real-world mathematical problems as shown in the situations below.

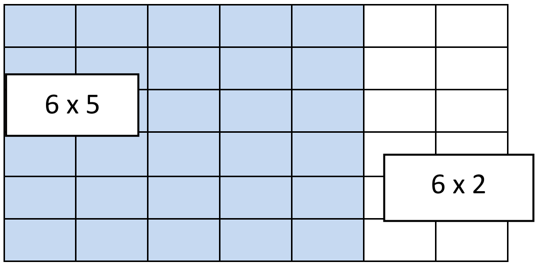
Example of tiling:

Drew wants to tile the bathroom floor using 1 foot tiles. How many square foot tiles will he need?



1 square foot = ☐

This standard also extends students' work with the **distributive property**. For example, in the picture below the area of a shape that is 6 by 7 can be determined by finding the area of the 6 x 5 section and the 6 x 2 section and then adding the two products together.

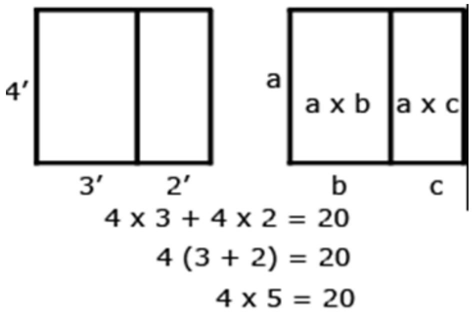


Students should tile areas of rectangles, determine the area, record the length and width of the rectangle, investigate the patterns in the numbers, and eventually uncover that the area can be determined by multiplying the length by the width.

Example:

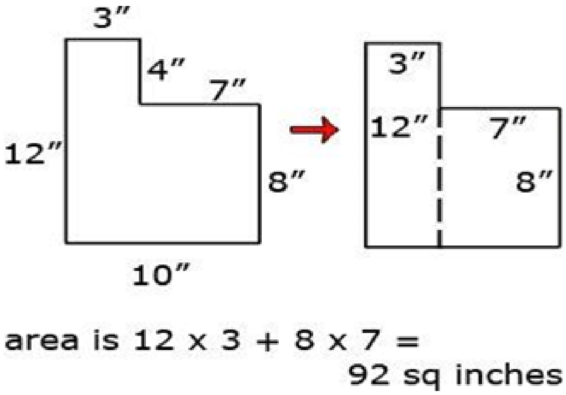
Joe and John made a poster that was 4ft. by 3ft. Melisa and Debbie made a poster that was 4ft. by 2ft. They placed their posters on the wall side-by-side so that that there was no space between them. How much area will the two posters cover?

Students use pictures, words, and numbers to explain their understanding of the distributive property in this context.



Example:

Students can decompose a rectilinear figure into different rectangles. They find the area of the figure by adding the areas of each of the rectangles together.



Instructional Strategies: See 3.MD.6

Tools/Resources:

See: for detailed information in:

- Learning Progressions-Measurement and Data (measurement part)
- Learning Progressions-Measurement and Data (data part)

Illustrative Math Tasks:

- Finding the Area of Polygons
- Three Hidden Rectangles
- India's Bathroom Tiles
- Introducing the Distributive Property

See: "Playful Puppies", NCSM, Great Tasks for Mathematics K-5, (2013)

NRICH mathematics

- Tiling

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **3rd Grade**. Scroll down to 3.MD.7 to access resources specifically for this standard.

Domain: Measurement and Data (MD)

● ***Cluster D:*** *Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures*.

Standard: 3.MD.9

Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters. **(3.MD.8)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected to:

- Measure and estimate lengths in standard units. Grade 2.MD.1 through 2.MD.4
- Relate addition and subtraction to length. Grade 2.MD.5 & 2.MD.6

Explanation and Examples:

Students develop an understanding of the concept of perimeter by walking around the perimeter of a room, using rubber bands to represent the perimeter of a plane figure on a geoboard, or tracing around a shape on an interactive whiteboard. They find the perimeter of objects; use addition to find perimeters; and recognize the patterns that exist when finding the sum of the lengths and widths of rectangles.

Students use geoboards, tiles, and graph paper to find all the possible rectangles that have a given perimeter (e.g., find the rectangles with a perimeter of 14 cm.) They record all the possibilities using dot or graph paper, compile the possibilities into an organized list or a table, and determine whether they have all the possible rectangles.

Given a perimeter and a length or width, students use objects or pictures to find the missing length or width. They justify and communicate their solutions using words, diagrams, pictures, numbers, and an interactive whiteboard.

Students use geoboards, tiles, graph paper, or technology to find all the possible rectangles with a given area (e.g. find the rectangles that have an area of 12 square units.) They record all the possibilities using dot or graph paper, compile the possibilities into an organized list or a table, and determine whether they have all the possible rectangles. Students then investigate the perimeter of the rectangles with an area of 12.

See example shown below:

Area	Length	Width	Perimeter
12 sq. in.	1 in.	12 in.	26 in.
12 sq. in.	2 in.	6 in.	16 in.
12 sq. in	3 in.	4 in.	14 in.
12 sq. in	4 in.	3 in.	14 in.
12 sq. in	6 in.	2 in.	16 in.
12 sq. in	12 in.	1 in.	26 in.

The patterns in the table allow the students to identify the factors of 12, connect the results to the commutative property, and discuss the differences in perimeter within the same area. This chart can also be used to investigate rectangles with the same perimeter. (Remember, squares are rectangles and should be a part of the investigation.)

Instructional Strategies:
Students have created rectangles when they were finding the area of rectangles and connecting them to using arrays in the multiplication of whole numbers.

To explore finding the perimeter of a rectangle, have students use twine or string (something that doesn't stretch).

- They should measure the twine and create a rectangle before cutting it into four pieces.
- Have the student make four pieces so that there are two pieces of one length and two pieces of a longer or shorter length.
- Students should be able to make the connection that perimeter is the total distance around the rectangle.

Geoboards can be used to find the perimeter of rectangles also. Provide students with different perimeters and have them create the rectangles on the geoboards. Have students share their rectangles with the class. Have discussions about how different rectangles can have the same perimeter with different side lengths.

Students experienced measuring lengths using inches and centimeters in Grade 2. They also related addition to length and wrote equations with a symbol for the unknown to represent a problem.

- Once students know how to find the perimeter of a rectangle, they can find the perimeter of rectangular-shaped objects in their environment.
- They can use appropriate measuring tools to find lengths of rectangular-shaped objects in the classroom.
- Present problems situations involving perimeter, such as finding the amount of fencing needed to enclose a rectangular shaped park, or how much ribbon is needed to decorate the edges of a picture frame.
- Present problem situations in which the perimeter and two or three of the side lengths are known, requiring students to find the unknown side length.

Students need to recognize when a problem situation requires a solution relates to the perimeter or the area and explain how they know.

They should have experience with understanding area concepts when they recognize it as an attribute of plane figures. They should also investigate that when plane figures are covered without gaps by *n* unit squares, the area of the figure is *n* square units.

Students need to explore how measurements are affected when one attribute to be measured is held constant and the other is changed. Using square tiles, students can discover that the area of rectangles may be the same, but the perimeter of the rectangles varies. Geoboards can also be used to explore this same concept.

Resources/Tools:

Illustrative Mathematics tasks:

- 3.MD Shapes and their Insides

NRICH mathematics

- Area and Perimeter lesson
- All About Area and Perimeter lessons
- Dicey Perimeter and Dicey Area game

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **3rd Grade**. Scroll down to 3.MD.8 to access resources specifically for this standard.

Common Misconceptions:

Students think that when they are presented with a drawing of a rectangle with only two of the side lengths shown or a problem situation with only two of the side lengths provided, these are the only dimensions they should add to find the perimeter. Encourage students to include the appropriate dimensions on the other sides of the rectangle. With problem situations, encourage students to make a drawing to represent the situation in order to find the perimeter.

Domain: Geometry (G)

◆ ***Cluster A:*** *Reason with shapes and their attributes.*

Standard: 3.G.1

Understand that shapes in different categories *(e.g. rhombuses, rectangles, trapezoids, kites and others)* may share attributes *(e.g. having four sides),* and that the shared attributes can define a larger category *(e.g. quadrilaterals).* Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories. Refer to inclusive definitions noted in the glossary. **(3.G.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected to:

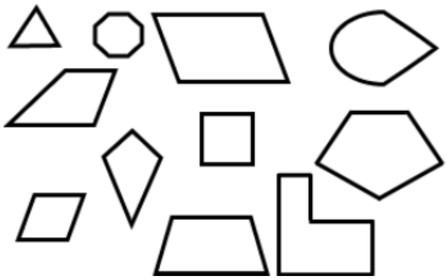
- Reason with shapes and their attributes. (Grade 2.G.1 through 2.G.3)

Explanation and Examples:

In second grade, students identify and draw triangles, quadrilaterals, pentagons, and hexagons. Third graders build on this experience and further investigate **quadrilaterals** (technology may be used during this exploration).

Students recognize shapes that are and are not quadrilaterals by examining the properties of the geometric figures. Definitions play a key role in determining if a shape is categorized appropriately or not.

Students conceptualize that a quadrilateral must be a closed figure with four straight sides AND they begin to notice characteristics of the angles and the relationship between opposite sides. Students should be encouraged to provide details and use proper vocabulary when describing the properties of quadrilaterals. They sort geometric figures (*see examples below*) and identify squares, rectangles, and rhombuses as quadrilaterals.



Students should classify shapes by attributes and drawing shapes that fit specific categories.
For example, parallelograms include: squares, rectangles, rhombi, or other shapes that have two pairs of parallel sides. Also, the broad category **quadrilaterals** include all types of parallelograms, trapezoids and other four-sided figures.

Example:

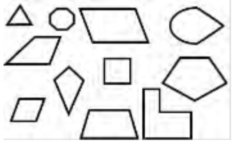
Draw a picture of a quadrilateral. Draw a picture of a rhombus.
How are they alike? How are they different?
Is a quadrilateral a rhombus? Is a rhombus a quadrilateral? Justify your thinking.

Instructional Strategies: (3.G.1 through 3.G.2)

In earlier grades, students have experiences with informal reasoning about particular shapes through sorting and classifying using the geometric attributes of the shapes. Students have built and drawn shapes given the number of faces, number of angles and number of sides.

The focus now is on identifying and describing properties of two-dimensional shapes in more precise ways using properties that are shared rather than the appearances of individual shapes. These properties allow for generalizations of all shapes that fit a particular classification.

Development in focusing on the identification and description of shapes' properties should include examples and non-examples, as well as examples and non-examples of shapes drawn by students ~~of shapes~~ in a particular category. For example, students could start with identifying shapes with **right angles**. An explanation as to why the remaining shapes do not fit this category should be discussed. Students should determine common characteristics of the remaining shapes.



Resources/Tools:

See Geometry Learning Progressions for detailed information:

Georgia Department of Education:

- "3-D Detectives" - Students identify, describe and illustrate plane and solid figures according to geometric properties.
- "What's In A Name" - Students describe and classify plane figures (triangles, square, rectangle, trapezoid, quadrilateral, pentagon, hexagon, and irregular polygonal shapes) by the number of edges, vertices and angles.

Visit K-5 Math Teaching Resources click on **Geometry**, then on **3rd Grade**. Scroll down to 3.G.1 to access resources specifically for this standard.

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- Shape Up

Common Misconceptions: (3.G.1 through 3.G.2)

Students may identify a square as a "non-rectangle" or a "non-rhombus" based on limited images they see. They do not recognize that a square is a rectangle. They may list properties of each shape separately, but not see the interrelationships between the shapes. Continually refer students back to the definitions of each of the shape categories. Then ask – *Does it fit the definition in all aspects? If it does, then it is in that category.*

Use straws to make four congruent figures and have students change the angles to see the relationships between a rhombus and a square. As students develop definitions for these shapes, relationships between the properties will be understood.

Domain: Geometry (G)

◆ ***Cluster A:*** *Reason with shapes and their attributes.*

Standard: 3.G.2

Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as* $\frac{1}{4}$ *of the area of the shape.* **(3.G.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

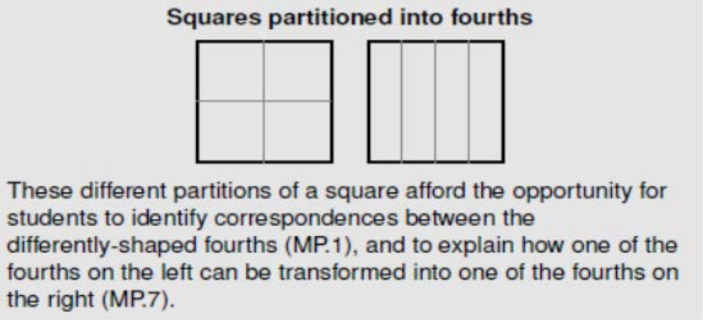
Connections: See 3.G.1

Explanation and Examples:

This standard builds on students' work with fractions and area. Students are responsible for partitioning shapes into halves, thirds, fourths, sixths and eighths.

Example:

These figures are partitioned/divided into four equal parts. Each part is $\frac{1}{4}$ of the total area of the figure.

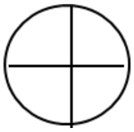


Squares partitioned into fourths

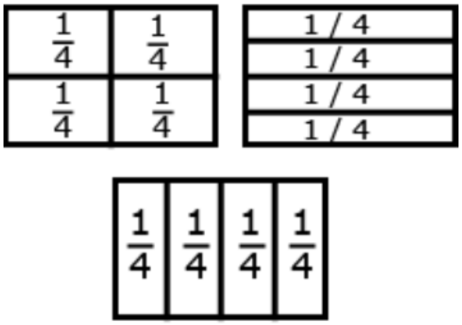
These different partitions of a square afford the opportunity for students to identify correspondences between the differently-shaped fourths (MP.1), and to explain how one of the fourths on the left can be transformed into one of the fourths on the right (MP.7).

Examples:

This figure was partitioned/divided into four equal parts. Each part is $\frac{1}{4}$ of the total area of the figure.

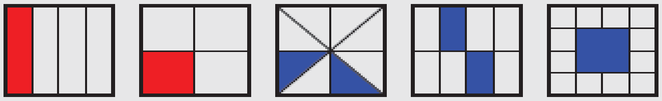


Given a shape, students partition it into equal parts, recognizing that these parts all have the same area. They identify the fractional name of each part and are able to partition a shape into parts with equal areas in several different ways.



Given a shape, students partition it into equal parts, recognizing that these parts all have the same area. They identify the fractional name of each part and are able to partition a shape into parts with equal areas in several different ways.

Area representations of $\frac{1}{4}$

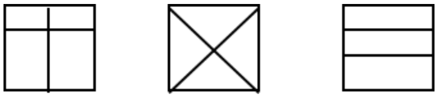


In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.

Learning Progressions:
Number & Operations-Fractions 3-5

Instructional Strategies: See 3.G.1.
In Grade 2, students partitioned rectangles into two, three or four equal shares, recognizing that the equal shares need not have the same shape. They described the shares using words such as, halves, thirds, half of, a third of, etc., and described the whole as two halves, three thirds or four fourths.

In Grade 4, students will partition shapes into parts with equal areas (the spaces in the whole of the shape). These equal areas need to be expressed as unit fractions of the whole shape, i.e., describe each part of a shape partitioned into four parts as $\frac{1}{4}$ of the area of the shape. 2 of the 3 shapes here don't represent 4 equal parts. Why are they here?



Have students draw different shapes and see how many ways they can partition the shapes into parts with equal area.

Tools/Resources:

Illustrative Mathematics tasks:

- 3.G Geometric pictures of one half
- 3.G Representing Half of a Circle
- 3.MD, 3.G, 3.NF Halves, thirds, and sixths

Visit K-5 Math Teaching Resources click on **Geometry**, then on **3rd Grade**. Scroll down to 3.G.2 to access resources specifically for this standard.

APPENDIX: TABLE 1. Common Addition and Subtraction Situations

Shading taken from OA progression

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Taken from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3
Put Together/ Take Apart[2]	**Total Unknown**	**Addend Unknown**	**Both Addends Unknown[1]**
	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2
Compare[3]	**Difference Unknown**	**Bigger Unknown**	**Smaller Unknown**
	(“How many more?” version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? (“How many fewer?” version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with “more”): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with “more”): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

Blue shading indicates the four Kindergarten problem subtypes. Students in grades 1 and 2 work with all subtypes and variants (blue and green). Yellow indicates problems that are the difficult four problem subtypes or variants that students in Grade 1 work with but do not need to master until Grade 2.

[1] These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.

[2] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

[3] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.

TABLE 2. Common Multiplication and Division Situations

Grade level identification of introduction of problem situations taken from OA progression

	Unknown Product	**Group Size Unknown** **("How many in each group?" Division)**	**Number of Groups Unknown** **("How many groups?" Division)**
	3 x 6 = ?	3 x ? = 18; 18 ÷ 3 = ?	? x 6 = 18; 18 ÷ 6 = ?
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays[4], Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	a × b = ?	a × ? = p, and p ÷ a = ?	? × b = p, and p ÷ b = ?

Multiplicative compare problems appear first in Grade 4 (green), with whole number values and with the "times as much" language from the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit language change the subject of the comparing sentence ("A red hat costs n times as much as the blue hat" results in the same comparison as "A blue hat is 1/n times as much as the red hat" but has a different subject.)

TABLE 3. The Properties of Operations

Name of Property	Representation of Property	Example of Property, Using Real Numbers
Properties of Addition		
Associative	$(a+b)+c=a+(b+c)$	$(78+25)+75=78+(25+75)$
Commutative	$a+b=b+a$	$2+98=98+2$
Additive Identity	$a+0=a$ and $0+a=a$	$9875+0=9875$
Additive Inverse	For every real number a, there is a real number $-a$ such that $a+-a=-a+a=0$	$-47+47=0$
Properties of Multiplication		
Associative	$(a\times b)\times c=a\times(b\times c)$	$(32\times 5)\times 2=32\times(5\times 2)$
Commutative	$a\times b=b\times a$	$10\times 38=38\times 10$
Multiplicative Identity	$a\times 1=a$ and $1\times a=a$	$387\times 1=387$
Multiplicative Inverse	*For every real number a,* $a\neq 0$, *there is a real number* $\frac{1}{a}$ *such that* $a\times\frac{1}{a}=\frac{1}{a}\times a=1$	$\frac{8}{3}\times\frac{3}{8}=1$
Distributive Property of Multiplication over Addition		
Distributive	$a\times(b+c)=a\times b+a\times c$	$7\times(50+2)=7\times 50+7\times 2$

(Variables a, b, and c represent real numbers.)

Excerpt from NCTM's ***Developing Essential Understanding of Algebraic Thinking****, grades 3-5 p. 16-17*

TABLE 4. The Properties of Equality

Name of Property	Representation of Property	Example of property
Reflexive Property of Equality	$a = a$	$3{,}245 = 3{,}245$
Symmetric Property of Equality	$If\ a = b, then\ b = a$	$2 + 98 = 90 + 10, then\ 90 + 10 = 2 + 98$
Transitive Property of Equality	$If\ a = b\ and\ b = c, then\ a = c$	$If\ 2 + 98 = 90 + 10\ and\ 90 + 10 = 52 + 48$ $then$ $2 + 98 = 52 + 48$
Addition Property of Equality	$If\ a = b, then\ a + c = b + c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{3}{5}$
Subtraction Property of Equality	$If\ a = b, then\ a - c = b - c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} - \frac{1}{5} = \frac{2}{4} - \frac{1}{5}$
Multiplication Property of Equality	$If\ a = b, then\ a \times c = b \times c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \times \frac{1}{5} = \frac{2}{4} \times \frac{1}{5}$
Division Property of Equality	$If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \div \frac{1}{5} = \frac{2}{4} \div \frac{1}{5}$
Substitution Property of Equality	If $a = b$, then b may be substituted for a in any expression containing a.	$If\ 20 = 10 + 10$ $then$ $90 + 20 = 90 + (10 + 10)$

(Variables a, b, and c can represent any number in the rational, real, or complex number systems.)

TABLE 5. The Properties of Inequality

Exactly one of the following is true: $a < b, a = b, a > b$.

$If\ a > b\ and\ b > c\ then\ a > c.$

$If\ a > b, then\ b < a.$

$If\ a > b, then - a < -b.$

$If\ a > b, then\ a \pm c > b \pm c.$

$If\ a > b\ and\ c > 0, then\ a \times c > b \times c.$

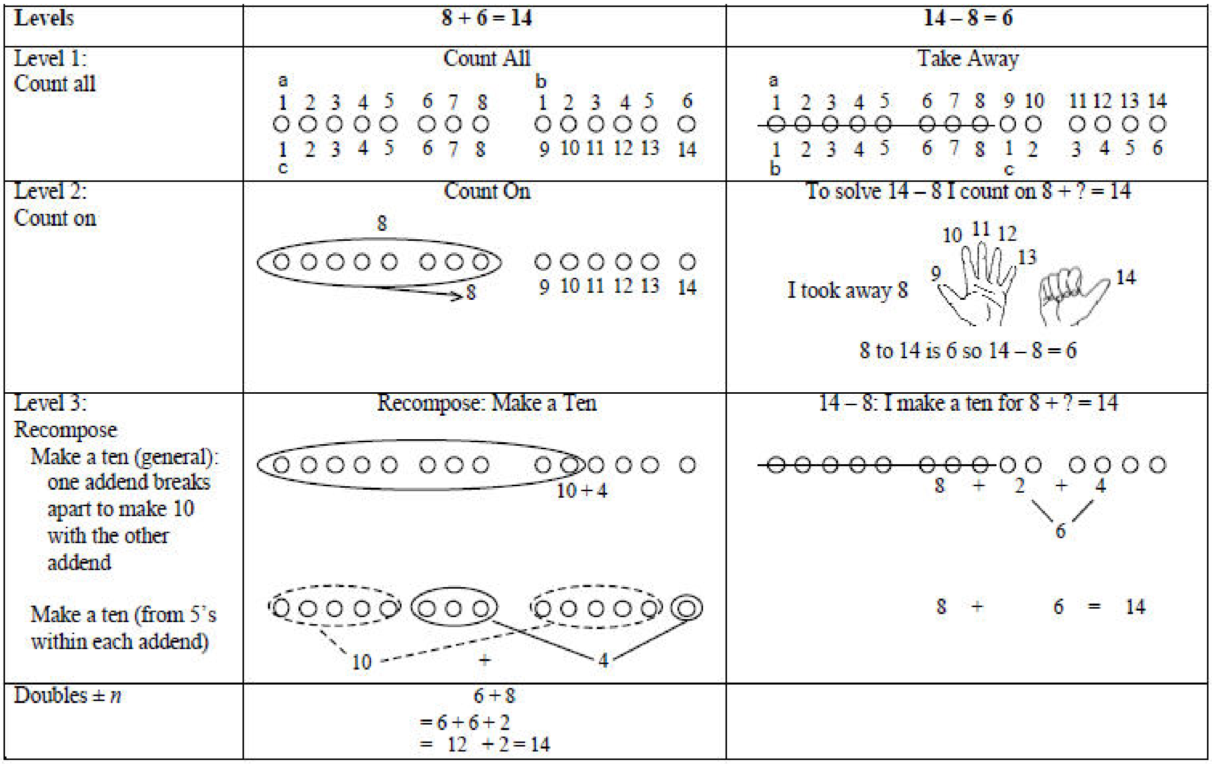
$If\ a > b\ and\ c < 0, then\ a \times c < b \times c.$

$If\ a > b\ and\ c > 0, then\ a \div c > b \div c.$

$If\ a > b\ and\ c < 0, then\ a \div c < b \div c.$

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

TABLE 6. Development of Counting in K-2 Children



Levels	8 + 6 = 14	14 – 8 = 6
Level 1: Count all	Count All a 1 2 3 4 5 6 7 8 b 1 2 3 4 5 6 ○○○○○ ○○○ ○○○○○ ○ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ~~○○○○○ ○○○~~ ○○ ○○○○ 1 2 3 4 5 6 7 8 1 2 3 4 5 6 b c
Level 2: Count on	Count On 8 ○○○○○ ○○○ ○○○○○ ○ 8 9 10 11 12 13 14	To solve 14 – 8 I count on 8 + ? = 14 I took away 8 9 10 11 12 13 14 8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten ○○○○○ ○○○ ○○○○○ ○ 10 + 4 ○○○○○ ○○○ ○○○○○ ○ 10 + 4	14 – 8: I make a ten for 8 + ? = 14 ~~○○○○○ ○○○~~ ○○ ○○○○ 8 + 2 + 4 6 8 + 6 = 14
Doubles ± *n*	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Beginning--A child can count very small collections (1-4) and understands that the last word tells "how many". Beyond small numbers the number words may be said without the lack of numerical understanding. This is often referred to as rote counting.

Level 1—The child uses one to one correspondence to find the number of objects in two sets. Even if the quantity is known for the first set, the child will start with the first set and continue counting on into the second set. The child begins the count with one. This also connects to Piaget's Hierarchical Inclusion – that in order to have 7 – you have to have 6, 5, 4, etc.

Level 2 – At this level the student begins the counting, starting with the known quantity of the first set and "counts on" from that point in the sequence to establish how many. This method is used to find the total in two sets. This counting is not rote. This level of counting requires several connections between cardinality and counting meanings of the number words.

Level 3 - At this level the student begins using known facts to solve for unknown facts. For example, the student uses "make a ten" where one addend breaks apart to make 10 with another addend OR a doubles plus/minus one strategy. Students begin to implicitly use the properties of operations.

Table 7. Cognitive Rigor Matrix/Depth of Knowledge (DOK)

The Kansas Math Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom)	**DOK Level 1** **Recall & Reproduction**	**DOK Level 2** **Basic Skills & Concepts**	**DOK Level 3** **Strategic Thinking & Reasoning**	**DOK Level 4** **Extended Thinking**
Remember	• Recall conversions, terms, facts			
Understand	• Evaluate an expression • Locate points on a grid or number on number line • Solve a one-step problem • Represent math relationships in words, pictures, or symbols	• Specify, explain relationships • Make basic inferences or logical predictions from data/observations • Use models/diagrams to explain concepts • Make and explain estimates	• Use concepts to solve non-routine problems • Use supporting evidence to justify conjectures, generalize, or connect ideas • Explain reasoning when more than one response is possible • Explain phenomena in terms of concepts	• Relate mathematical concepts to other content areas, other domains • Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	• Follow simple procedures • Calculate, measure, apply a rule (e.g., rounding) • Apply algorithm or formula • Solve linear equations • Make conversions	• Select a procedure and perform it • Solve routine problem applying multiple concepts or decision points • Retrieve information to solve a problem • Translate between representations	• Design investigation for a specific purpose or research question • Use reasoning, planning, and supporting evidence • Translate between problem & symbolic notation when not a direct translation	• Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	• Retrieve information from a table or graph to answer a question • Identify a pattern/trend	• Categorize data, figures • Organize, order data • Select appropriate graph and organize & display data • Interpret data from a simple graph • Extend a pattern	• Compare information within or across data sets or texts • Analyze and draw conclusions from data, citing evidence • Generalize a pattern • Interpret data from complex graph	• Analyze multiple sources of evidence or data sets
Evaluate			• Cite evidence and develop a logical argument • Compare/contrast solution methods • Verify reasonableness	• Apply understanding in a novel way, provide argument or justification for the new application
Create	• Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	• Generate conjectures or hypotheses based on observations or prior knowledge and experience	• Develop an alternative solution • Synthesize information within one data set	• Synthesize information across multiple sources or data sets • Design a model to inform and solve a practical or abstract situation

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