6/26/2018

2017 Kansas
Mathematics Standards

Flip Book
Kindergarten

This project used work created by the Departments of Education in Ohio, North Carolina, Georgia and resources created by Achieve the Core, EngageNY, Illustrative Mathematics, and NCTM.

About the Flip Books

This project attempts to organize some of the most valuable resources that help develop the intent, understanding and implementation of the 2017 Kansas Mathematics Standards. These documents provide a starting point for teachers and administrators to begin discussion and exploration into the standards. It is not the only resource to support implementation of the 2017 Kansas Mathematics Standards.

This project is built on the previous work started in the summer of 2012 from Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the "flip books." The "flip books" are based on a model that Kansas had for earlier standards; however, this edition specifically targets the Kansas Mathematics Standards that were adopted in the summer of 2017. These flip books incorporate the resources from other state departments of education, the mathematics learning progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KSDE Mathematics website at http://community.ksde.org/Default.aspx?tabid=5646 and will continue to undergo changes periodically. When significant changes/additions are implemented, the modifications will be posted and dated.

For questions or comments about the flipbooks, please contact Melissa Fast at the Kansas State Department of Education – mfast@ksde.org.

Planning Advice - Focus on the Clusters

The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)

Not all standards should have the same instructional emphasis. Some groups of standards require a greater emphasis than others. In order to be intentional and systematic, priorities need to be set for planning, instruction, and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "while the remaining content is limited in scope." 4) a "lower" priority does not imply exclusion of content, but is usually intended to be taught in conjunction with or in support of one of the major clusters.

"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path, if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In order to accomplish this, educators need to think about "grain size" when planning instruction. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Phil Daro (*Teaching Chapters, Not Lessons—Grain Size of Mathematics*), strands are too vague and too large a grain size, while lessons are too small a grain size. Units or chapters produce about the right "grain size". In the planning process educators should attend to the clusters, and think of the standards as the ingredients of a cluster. Coherence of mathematical ideas and concepts exists at the cluster level across grades.

A caution--Grain size is important but can result in conversations that do not advance the intent of this structure. Extended discussions among teachers where it is argued for "2 days" instead of "3 days" on a topic because it is a lower priority can detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, focusing on lessons can provide too narrow a view which compromises the coherence value of closely related standards.

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics presents Phil Daro further explaining grain size and the importance of it in the planning process. (Click on photo to view video.)

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they determine distribution of time for both planning and instruction, helping to assure that students really understand mathematics before moving on. Each cluster has been given a priority level. As professional educators begin planning, developing and writing units, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level influenced by the work of Jason Zimba. The three levels are referred to as — **Major, Supporting** and **Additional**. Zimba suggests that about 70% of instruction should relate to the **Major** clusters. The lower two priorities (**Supporting** and **Additional**) can work together by supporting the **Major** priorities. You can find the grade Level Focus Documents for the 2017 Kansas Math Standards at: http://community.ksde.org/Default.aspx?tabid=6340.

Recommendations for Cluster Level Priorities

Appropriate Use:

- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis.
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possible quality; the additional work of the grade should support the major priorities and not detract from them.
- Set priorities for other implementation efforts such as staff development, new curriculum development, and revision of existing formative or summative testing at the state, district or school level.

Things to Avoid:

- Neglecting any of the material in the standards. Seeing Supporting and Additional clusters as optional.
- Sorting clusters (from Major to Supporting to Additional) and then teaching the clusters in order. This would remove the coherence of mathematical ideas and create missed opportunities to enhance the major work of the grade with the other clusters.
- Using the cluster headings as a replacement for the actual standards. All features of the standards matter—from the practices to surrounding text, including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise the coherence of the standards (grain size).

Mathematics Teaching Practices

(High Leverage Teacher Actions)

[National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.]

The eight Mathematics Teaching Practices should be the foundation for mathematics instruction and learning. This framework was informed by over twenty years of research and presented in *Principles to Actions* by the National Council of Teachers of Mathematics (NCTM). If teachers are guided by this framework, they can move "toward improved instructional practice" and support "one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students" (NCTM, 2014, p. 12).

1. **Establish mathematics goals to focus learning.**
   Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2. **Implement tasks that promote reasoning and problem solving.**
   Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3. **Use and connect mathematical representations.**
   Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4. **Facilitate meaningful mathematical discourse.**
   Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
5. **Pose purposeful questions.**
   Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
6. **Build procedural fluency from conceptual understanding.**
   Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7. **Support productive struggle in learning mathematics.**
   Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8. **Elicit and use evidence of student thinking.**
   Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Standards for Mathematical Practice in Kindergarten

The Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students grades K-12. Below are a few examples of how these Practices may be integrated into tasks that Kindergarten students complete.

Practice	Explanation and Example
1) Make sense of problems and persevere in solving them.	Mathematically proficient students in Kindergarten begin to develop effective dispositions toward problem solving. In rich settings in which informal and formal possibilities for solving problems are numerous, young children develop the ability to focus attention, test hypotheses, take reasonable risks, remain flexible, try alternatives, exhibit self-regulation, and persevere (Copley, 2010). Using both verbal and nonverbal means, kindergarten students begin to explain to themselves and others the meaning of a problem, look for ways to solve it, and determine if their thinking makes sense or if another strategy is needed. As the teacher uses thoughtful questioning and provides opportunities for students to share thinking, kindergarten students begin to reason as they become more conscious of what they know and how they solve problems.
2) Reason abstractly and quantitatively.	Mathematically proficient students in Kindergarten begin to use numerals to represent specific amounts (quantities). For example, a student may write the numeral “11” to represent an amount of eleven objects counted, select the correct number card “17” to follow “16” on the calendar, or build a pile of counters depending on the number selected. In addition, kindergarten students begin to draw pictures, manipulate objects, use diagrams or charts, etc. to express quantitative ideas such as a joining situation (Mary has 3 bears. Juanita gave her 1 more bear. How many bears does Mary have altogether?), or a separating situation (Mary had 5 bears. She gave some to Juanita. Now she has 3 bears. How many bears did Mary give Juanita?). Using the language developed through numerous joining and separating scenarios, kindergarten students begin to understand how symbols (+, -, =, ≠) are used to represent quantitative ideas in a written format.
3) Construct viable arguments and critique the reasoning of others.	In Kindergarten, mathematically proficient students begin to clearly express, explain, organize and consolidate their math thinking using both verbal and written representations. Through opportunities that encourage exploration, discovery, and discussion, kindergarten students begin to learn how to express opinions, become skillful at listening to others, describe their reasoning and respond to others’ thinking and reasoning. They begin to develop the ability to reason and analyze situations as they consider questions such as, “Are you sure…?” , “Do you think that would happen all the time…?”, and “I wonder why…?”
4) Model with mathematics.	Mathematically proficient students in Kindergarten begin to experiment with representing real-life problem situations in multiple ways such as with numbers, words (mathematical language), drawings, objects, acting out, charts, lists, and number sentences. For example, when making toothpick designs to represent the various combinations of the number “5”, the student writes the numerals for the various parts (such as “4” and “1”) or selects a number sentence that represents that particular situation (such as 5 = 4 + 1)*. Kindergartners experiment with representing problem situations in multiple ways including numbers, words (mathematical language), drawing pictures, using objects, acting out, making a chart or list, creating equations, etc. Students need many opportunities to connect the different representations and explain the connections. *“Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten in encouraged, but it is not required”. However, it is not until First Grade when “Understand the meaning of the equal sign” is an expectation (1.OA.7).

5) Use appropriate tools strategically.	In Kindergarten, mathematically proficient students begin to explore various tools and use them to investigate mathematical concepts. Through multiple opportunities to examine materials, they experiment and use both concrete materials (e.g. 3-dimensional solids, connecting cubes, ten frames, number balances) and technological materials (e.g., virtual manipulatives, calculators, and interactive websites) to explore mathematical concepts. Based on these experiences, they become able to decide which tools may be helpful to use depending on the problem or task. For example, when solving the problem, "There are 4 dogs in the park. 3 more dogs show up in the park. How many dogs are in the park?", students may decide to act it out using counters and a story mat; draw a picture; or use a handful of cubes.
6) Attend to precision.	Mathematically proficient students in Kindergarten begin to express their ideas and reasoning using words. As their mathematical vocabulary increases due to exposure, modeling, and practice, kindergarteners become more precise in their communication, calculations, and measurements. In all types of mathematical tasks, students begin to describe their actions and strategies more clearly, understand and use grade-level appropriate vocabulary accurately, and begin to give precise explanations and reasoning regarding their process of finding solutions. For example, a student may use color words (such as blue, green, light blue) and descriptive words (such as small, big, rough, smooth) to accurately describe how a collection of buttons is sorted.
7) Look for and make use of structure.	Mathematically proficient students in Kindergarten begin to look for patterns and structures in the number system and other areas of mathematics. For example, when searching for triangles around the room, kindergarteners begin to notice that some triangles are larger than others or come in different colors- yet they are all triangles. While exploring the part-whole relationships of a number using a number balance, students begin to realize that 5 can be broken down into sub-parts, such as 4 and 1 or 4 and 2, and still remain a total of 5. Another example might be that the students represent a story about children moving between two classrooms: the number of children in each classroom is an addend; the total number of children in the two classrooms is the sum. When some students move from one classroom to the other, the number of students in each classroom changes by that amount—one addend decreases by some amount and the other addend increases by that same amount—but the total number of students does not change.
8) Look for and express regularity in repeated reasoning.	In Kindergarten, mathematically proficient students begin to notice repetitive actions in geometry, counting, comparing, etc. For example, a kindergartener may notice that as the number of sides increase on a shape, a new shape is created (triangle has 3 sides, a rectangle has 4 sides, a pentagon has 5 sides, a hexagon has 6 sides). When counting out loud to 100, kindergartners may recognize the pattern 1-9 being repeated for each decade (e.g., Seventy-ONE, Seventy-TWO, Seventy-THREE… Eighty-ONE, Eighty-TWO, Eighty-THREE…etc.). When joining one more cube to a pile, the child may realize that the new amount is the next number in the count sequence. When using a Ten Frame with 8 counters, they notice there are 2 spaces; or with 4 counters on the Ten Frame, they notice there are 6 spaces. As they look for and explain their reasoning they continually ask themselves, "Does this make sense"?

Adapted from the work of the State Department of Education of North Carolina.

Implementing Standards for Mathematical Practice

This guide was created to help educators implement these standards into their classroom instruction. These are the practices for the **students,** and the teacher can assist students in using them efficiently and effectively.

#1 – Make sense of problems and persevere in solving them.

Summary of this Practice:

- Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, "Does this make sense?"
- Understand various approaches to solutions.

Student Actions	Teacher Actions
• Actively engage in solving problems and thinking is visible (doing mathematics vs. following steps or procedures with no understanding). • Relate current "situation" to concepts or skills previously learned, and checking answers using different methods. • Monitor and evaluate their own progress and change course when necessary. • Always ask, "Does this make sense?" as they are solving problems.	• Allow students time to initiate a plan; using question prompts as needed to assist students in developing a pathway. • Constantly ask students if their plans and solutions make sense. • Question students to see connections to previous solution attempts and/or tasks to make sense of the current problem. • Consistently ask students to defend and justify their solution(s) by comparing solution paths.

What questions develop this Practice?

- How would you describe the problem in your own words? How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem? Describe the relationship between the quantities.
- Describe what you have already tried. What might you change? Talk me through the steps you've used to this point.
- What steps in the process are you most confident about? What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

What are the characteristics of a good math task for this Practice?

- Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available.
- Encourages multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

#2 – Reason abstractly and quantitatively.

Summary of this Practice:

- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attend to the meaning of quantities, not just how to compute them.

Student Actions	Teacher Actions
• Use varied representations and approaches when solving problems. • Represent situations symbolically and manipulating those symbols easily. • Give meaning to quantities (not just computing them) and making sense of the relationships within problems.	• Ask students to explain the meaning of the symbols in the problem and in their solution. • Expect students to give meaning to all quantities in the task. • Question students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

What questions develop this Practice?

- What do the numbers used in the problem represent? What is the relationship of the quantities?
- How is_____related to______?
- What is the relationship between _____ and _____?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might you use to find a solution?
- How did you decide that you needed to use _____? Could we have used another operation or property to solve this task? Why or why not?

What are the characteristics of a good math task for this Practice?

- Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

#3 – Construct viable arguments and critique the reasoning of others.

Summary of this Practice:

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

Student Actions	Teacher Actions
• Make conjectures and exploring the truth of those conjectures. • Recognize and use counter examples. • Justify and defend all conclusions and using data within those conclusions. • Recognize and explain flaws in arguments, which may need to be demonstrated using objects, pictures, diagrams, or actions.	• Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems etc.), to support their reasoning. • Question students so they can tell the difference between assumptions and logical conjectures. • Ask questions that require students to justify their solution and their solution pathway. • Prompt students to respectfully evaluate peer arguments when solutions are shared. • Ask students to compare and contrast various solution methods • Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.)

What questions develop this Practice?

- What mathematical evidence would support your solution? How can we be sure that...? How could you prove that...?
- Will it still work if...?
- What were you considering when...? How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn’t it work? Would it ever work? Why or why not?
- What is the same and what is different about...? How could you demonstrate a counter-example?

What are the characteristics of a good math task for this Practice?

- Structured to bring out multiple representations, approaches, or error analysis.
- Embeds discussion and communication of reasoning and justification with others.
- Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- Expects students to give feedback and ask questions of others’ solutions.

#4 – Model with mathematics.

Summary of this Practice:

- Understand reasoning quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the math they know to solve problems in everyday life.
- Simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, "How can I represent this mathematically?"

Student Actions	Teacher Actions
• Apply mathematics to everyday life. • Write equations to describe situations. • Illustrate mathematical relationships using diagrams, data displays, and/or formulas. • Identify important quantities and analyzing relationships to draw conclusions.	• Demonstrate and provide students experiences with the use of various mathematical models. • Question students to justify their choice of model and the thinking behind the model. • Ask students about the appropriateness of the model chosen. • Assist students in seeing and making connections among models.

What questions develop this Practice?

- What number model could you construct to represent the problem?
- How can you represent the quantities?
- What is an equation or expression that matches the diagram..., number line..., chart..., table...?
- Where did you see one of the quantities in the task in your equation or expression?
- What math do you know that you could use to represent this situation?
- What assumptions do you have to make to solve the problem?
- What formula might apply in this situation?

What are the characteristics of a good math task for this Practice?

- Structures represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.

#5 – Use appropriate tools strategically.

Summary of this Practice:

- Use available tools recognizing the strengths and limitations of each.
- Use estimation and other mathematical knowledge to detect possible errors.
- Identify relevant external mathematical resources to pose and solve problems.
- Use technological tools to deepen their understanding of mathematics.
- Use mathematical models for visualize and analyze information

Student Actions	Teacher Actions
• Choose tools that are appropriate for the task. • Know when to use estimates and exact answers. • Use tools to pose or solve problems to be most effective and efficient.	• Demonstrate and provide students experiences with the use of various math tools. A variety of tools are within the environment and readily available. • Question students as to why they chose the tools they used to solve the problem. • Consistently model how and when to estimate effectively, and requiring students to use estimation strategies in a variety of situations. • Ask student to explain their mathematical thinking with the chosen tool. • Ask students to explore other options when some tools are not available.

What questions develop this practice?

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem? What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...?
- What can using a_____show us that _____may not?
- In what situations might it be more informative or helpful to use...?

What are the characteristics of a good math task for this Practice?

- Lends itself to multiple learning tools. (Tools may include; concrete models, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.)
- Requires students to determine and use appropriate tools to solve problems.
- Asks students to estimate in a variety of situations:
    - a task when there is no need to have an exact answer
    - a task when there is not enough information to get an exact answer
    - a task to check if the answer from a calculation is reasonable

#6 – Attend to precision.

Summary of this Practice:

- Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.
- Understand meanings of symbols used in mathematics and can label quantities appropriately.
- Express numerical answers with a degree of precision appropriate for the problem context.
- Calculate efficiently and accurately.

Student Actions	Teacher Actions
• Use mathematical terms, both orally and in written form, appropriately. • Use and understanding the meanings of math symbols that are used in tasks. • Calculate accurately and efficiently. • Understand the importance of the unit in quantities.	• Consistently use and model correct content terminology. • Expect students to use precise mathematical vocabulary during mathematical conversations. • Question students to identify symbols, quantities and units in a clear manner.

What questions develop this Practice?

- What mathematical terms apply in this situation? How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain...?
- How could you test your solution to see if it answers the problem?

What are the characteristics of a good math task for this Practice?

- Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- Expects students to use symbols appropriately.
- Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

#7 – Look for and make use of structure.

Summary of this Practice:

- Apply general mathematical rules to specific situations.
- Look for the overall structure and patterns in mathematics.
- See complicated things as single objects or as being composed of several objects.

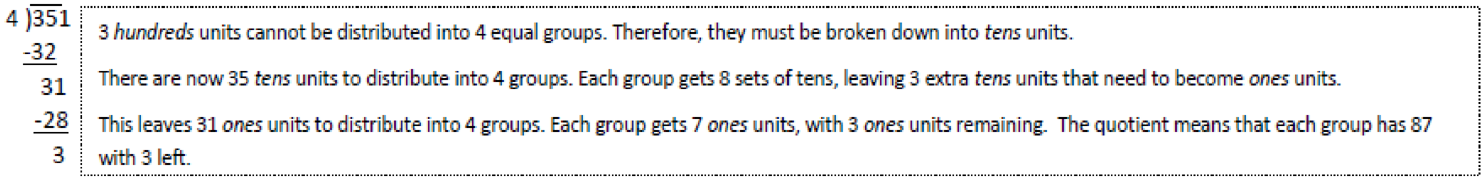
Student Actions	Teacher Actions
• Look closely at patterns in numbers and their relationships to solve problems. • Associate patterns with the properties of operations and their relationships. • Compose and decompose numbers and number sentences/expressions.	• Encourage students to look for something they recognize and having students apply the information in identifying solution paths (i.e. compose/decompose numbers and geometric figures, identify properties, operations, etc.) • Expect students to explain the overall structure of the problem and the big math idea used to solve the problem.

What questions develop this Practice?

- What observations do you make about...? What do you notice when...?
- What parts of the problem might you eliminate..., simplify...?
- What patterns do you find in...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one? How does this relate to...?
- In what ways does this problem connect to other mathematical concepts?

What are the characteristics of a good math task for this Practice?

- Requires students to look for the structure within mathematics in order to solve the problem. (i.e. – decomposing numbers by place value; working with properties; etc.)
- Asks students to take a complex idea and then identify and use the component parts to solve problems. i.e. Building on the structure of equal sharing, students connect the understanding to the traditional division algorithm. When “unit size” cannot be equally distributed, it is necessary to break down into a smaller “unit size”. (example below)



4)351
-32
31
-28
3

3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units.

There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units.

This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

- Expects students to recognize and identify structures from previous experience(s) and apply this understanding in a new situation. i.e. $7 \times 8 = (7 \times 5) + (7 \times 3)$ OR $7 \times 8 = (7 \times 4) + (7 \times 4)$ new situations could be, distributive property, area of composite figures, multiplication fact strategies.

#8 – Look for and express regularity in repeated reasoning.

Summary of this Practice:

- See repeated calculations and look for generalizations and shortcuts.
- See the overall process of the problem and still attend to the details.
- Understand the broader application of patterns and see the structure in similar situations.
- Continually evaluate the reasonableness of their intermediate results.

Student Actions	Teacher Actions
• Notice if processes are repeated and look for both general methods and shortcuts. • Evaluate the reasonableness of intermediate results while solving. • Make generalizations based on discoveries and constructing formulas when appropriate.	• Ask what math relationships or patterns can be used to assist in making sense of the problem. • Ask for predictions about solutions at midpoints throughout the solution process. • Question students to assist them in creating generalizations based on repetition in thinking and procedures.

What questions develop this Practice?

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true? How would we prove that...?
- What do you notice about...?
- What is happening in this situation? What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

What are the characteristics of a good math task for this Practice?

- Present several opportunities to reveal patterns or repetition in thinking, so students can make a generalization or rule.
- Requires students to see patterns or relationships in order to develop a mathematical rule.
- Expects students to discover the underlying structure of the problem and come to a generalization.
- Connects to a previous task to extend learning of a mathematical concept.

Critical Areas for COHERENCE in Mathematics in Kindergarten

More learning time in Kindergarten should be devoted to number than to other topic.

In Kindergarten, instructional time should focus on **three** critical areas:

1. **Representing and comparing whole numbers, initially with sets of object.**
   Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals. Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes. Students understand “teen” numbers are ten ones and some more ones.

2. **Understanding addition as putting together and adding to, and subtraction as taking apart and taking from.**
   Students begin to model simple joining and separating situations with sets of objects, or eventually with equations such as $5 + 2 = 7$ and $7 - 2 = 5$. (Kindergarten students should see addition and subtraction equations, and students writing of equations in kindergarten is encouraged, but it is **not** required.). Students apply effective strategies for counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away but are not expected to work above 10.

3. **Describing shapes and space.**
   Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

Dynamic Learning Maps

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Kansas Mathematics Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Kansas Mathematics Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

Growth Mindset

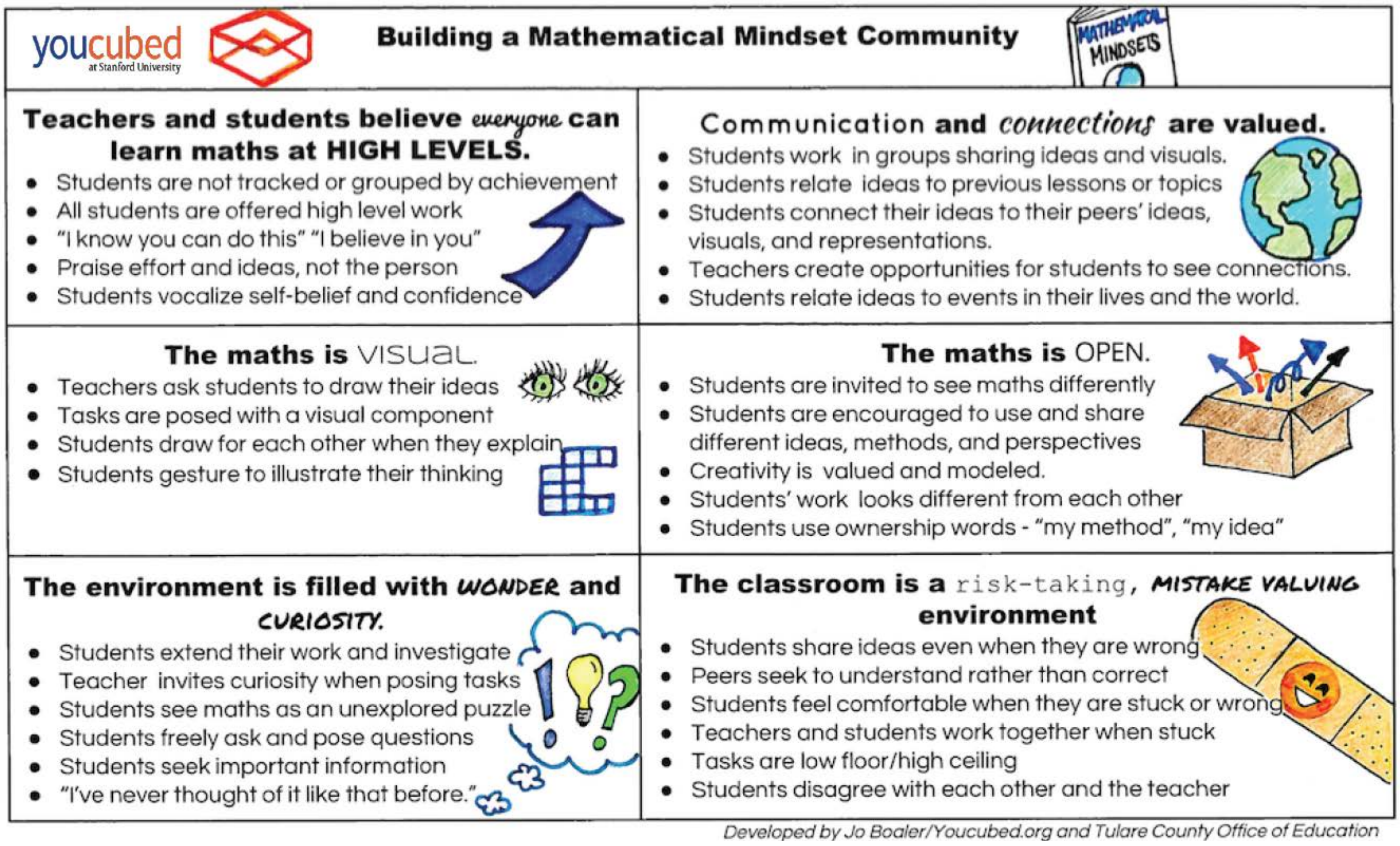
The term “growth mindset” comes from the groundbreaking work of Carol Dweck. She identified that everyone holds ideas about their own potential. Some people believe that their intelligence is more or less fixed in math – that you can do math or you can’t, while others believe they can learn anything and that their intelligence can grow.

In a fixed mindset, people believe their basic qualities, like their intelligence or talent, are simply fixed traits. They spend their time documenting their intelligence or talent instead of developing it. They also believe that talent alone creates success—without effort. Students with a fixed mindset are those who are more likely to give up easily.

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Students with a growth mindset are those who keep going even when work is hard, and who are persistent.

It is possible to change mindsets and to shift students’ mindsets from fixed to growth and cause higher mathematics achievement and success in life. Watch this short video to get a better understanding of what Growth Mindset is and the benefits it can bring our students.

You can find a variety of resources related to **Growth Mindset** at: http://community.ksde.org/Default.aspx?tabid=6383.



Building a Mathematical Mindset Community

Teachers and students believe *everyone* can learn maths at HIGH LEVELS.

- Students are not tracked or grouped by achievement
- All students are offered high level work
- “I know you can do this” “I believe in you”
- Praise effort and ideas, not the person
- Students vocalize self-belief and confidence

Communication and *connections* are valued.

- Students work in groups sharing ideas and visuals.
- Students relate ideas to previous lessons or topics
- Students connect their ideas to their peers’ ideas, visuals, and representations.
- Teachers create opportunities for students to see connections.
- Students relate ideas to events in their lives and the world.

The maths is VISUAL.

- Teachers ask students to draw their ideas
- Tasks are posed with a visual component
- Students draw for each other when they explain
- Students gesture to illustrate their thinking

The maths is OPEN.

- Students are invited to see maths differently
- Students are encouraged to use and share different ideas, methods, and perspectives
- Creativity is valued and modeled.
- Students’ work looks different from each other
- Students use ownership words - “my method”, “my idea”

The environment is filled with *WONDER* and *CURIOSITY.*

- Students extend their work and investigate
- Teacher invites curiosity when posing tasks
- Students see maths as an unexplored puzzle
- Students freely ask and pose questions
- Students seek important information
- “I’ve never thought of it like that before.”

The classroom is a risk-taking, *MISTAKE VALUING* environment

- Students share ideas even when they are wrong
- Peers seek to understand rather than correct
- Students feel comfortable when they are stuck or wrong
- Teachers and students work together when stuck
- Tasks are low floor/high ceiling
- Students disagree with each other and the teacher

Developed by Jo Boaler/Youcubed.org and Tulare County Office of Education

Kindergarten Content Standards Overview

Counting and Cardinality (K.CC)

A. Know number names and the count sequence.
CC.1 CC.2 CC.3

B. Count to tell the number of objects.
CC.4 CC.5

C. Compare numbers.
CC.6 CC.7

Operations and Algebraic Thinking (K.OA)

A. Understand addition as putting together and adding to and understand subtraction as taking apart and taking from.
OA.1 OA.2 OA.3
OA.4 OA.5

Number and Operations in Base Ten (K.NBT)

A. Work with numbers 11-19 to gain foundations for place value.
NBT.1

Measurement and Data (K.MD)

A. Describe and compare measurable attributes.
MD.1 MD.2

B. Classify objects and count the number of objects in each category.
MD.3

Geometry (K.G)

A. Identify and describe shapes.
G.1 G.2 G.3

B. Analyze, compare, create, and compose shapes.
G.4 G.5 G.6

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

▶ Major Clusters ◆ Supporting Clusters ● Additional Clusters

Domain: Counting and Cardinality (CC)

▶ **Cluster A:** Know number names and the count sequence.

Standard: K.CC.1

Count to 100 by ones and by tens and identify as a growth pattern. **(K.CC.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

See the Early Childhood Standards.

Explanation and Examples:

The emphasis of this standard is on the counting sequence and identifying this as a growth pattern. When counting by ones, students need to understand that the next number in the sequence is one more (which is a growth pattern). When counting by tens, the next number in the sequence is “ten more” (or one more group of ten which is another growth pattern).

Students are to rote count (verbal expression of numbers in sequence) by starting at one and counting up to 100. This objective does not require recognition of the numerals. While students will usually learn the counting words, they may not know the quantities attached to those words. This is not unusual. Teachers will want to connect the numerals to quantities so students can see that each number is a symbol representing a specific quantity.

Provide settings that connect mathematical language to the everyday lives of kindergarteners. Support students’ ability to make meaning and mathematize (*the process of seeing and focusing on the mathematical aspects and ignoring the non-mathematical aspects).*

Mathematizing in Kindergarten: Solving problems, Communicating or showing their thinking, Connecting and Representing Ideas. In the real world, help students see patterns, make connections and provide repeated experiences that give them time and opportunity to develop understanding and increase fluency. Encourage students to explain their reasoning by asking probing questions such as “How do you know?”, “How did you figure that out?”.

When counting orally, students will begin to recognize the patterns that exist from 1 to 100. They will also begin to recognize the patterns that exist when counting by 10s. Have students verbalize the patterns they hear.

Help them see patterns, make connections and provide repeated experiences that give students time and opportunities to develop understandings and increase fluency.

Games that require students to add on to a previous count to reach a goal number encourage developing this concept. Frequent and brief opportunities utilizing counting on and counting back are recommended. These concepts emerge over time and cannot be forced. As with many physical activities, counting will improve with practice and does not need to be perfect each time. It is much more important for all children to get frequent counting practice and watch and help one another, with help and correction from the teacher.

Tools /Resources

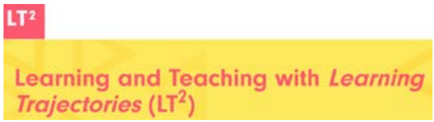
For detailed information, see: Learning Progression Counting and Cardinality

See Appendix Table 6: "Development of Counting"

Illustrative Mathematics tasks:

- K.CC Teens Go Fish
- K.CC Find The Numbers 0-5 or 5-10
- K.CC Assessing Sequencing Numbers
- K.CC Five by Two
- K.CC Choral Counting
- K.CC Counting Circles
- K.CC Assessing Counting Sequences Part I
- K.CC Counting by Tens

Dr. Douglas Clements and Dr. Julia Sarama are well-respected early numeracy educators and researchers. Their book, Learning and Teaching Early Math: The Learning Trajectories Approach, provides a fabulous learning trajectory for *Verbal and Object Counting* from the age of 1 to 7. These researchers explain that verbal counting is essential in developing quantitative thinking. They also have provided educators access to their online trajectories. Make sure you register to gain access to their developmental progressions and the activities and lessons to go with each stage of learning.



Visit **K-5 Math Teaching Resources** section showing the counting stages with corresponding games and activities. Click on Number and then select Kindergarten.

Consider using Number Paths instead of Number Lines for those students that have trouble with continuous units. Number Paths use discrete units and can be a bridge to the continuous number line.

Common Misconceptions:

Some students have difficulty in understanding that *zero* is a number. Ask students to write *0* and say *zero* to represent the number of items left when all objects have been taken away. Find instances for which the response would be *zero* in real-world settings to provide experiences for the concept of zero.

Number lines should display the number 0 instead of starting with the number 1. Students should begin to experience and understand that numbers can be identified by the distance from 0.

Using songs and rhymes can be beneficial as students learn to count, but care must be taken so the number words are separate from each other and that each is a counting word. (Avoid the "elemeno" effect some students learn from memorizing the alphabet.)

Domain: Counting and Cardinality (CC)

▶ **Cluster A**: Know number names and the count sequence.

Standard: K.CC.2

Count forward beginning from a given number within the known sequence (instead of having to begin at 1). **(K.CC.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

See the Early Childhood Standards.

Explanation and Examples:

The emphasis of this standard is on the counting sequence to 100. **K.CC.2** includes numbers 0-100. This asks for students to begin rote counting forward in a sequence from a number other than one (e.g.: given the number 4, the student would count, "4, 5, 6 . . . ."). This objective does not require recognition of numerals. It is focused on verbal counting or the rote number sequence.

Being able to count forward from any number is a prerequisite skill to "counting on" which students will work with in first grade.

Instructional Strategies:

Games that require students to add on to a previous count to reach a goal number encourage development of this concept. Frequent and brief opportunities utilizing *counting on* and *counting back* are recommended. **These concepts emerge over time and should not be forced**.

Tools/Resources

Illustrative Mathematics tasks:

- K.CC Assessing Counting Sequences Part II
- K.CC Number Line Up
- K.CC Start-Stop Counting
- K.CC Assessing Counting Sequences Part I
- K.CC Number After Bingo 1-15
- K.CC Pick a Number, Counting On
- K.CC "One More" Concentration

BetterLesson Plans Website: Pat the Pest Man - Count from Any Number

Visit Dr. Doug Clements' and Dr. Julie Sarama's website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

LT^2

Learning and Teaching with *Learning Trajectories* (LT^2)

Domain: Counting and Cardinality (CC)

▶ **Cluster A:** Know number names and the count sequence.

Standard: K.CC.3

Read and write numerals from 0 to 20. **(K.CC.3)**

Suggested Standards for Mathematical Practice (MP

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This standard is connected to:

- K.CC.4
- K.NBT.1
- K.MD.3

Explanation and Examples:

Like counting to 100 by ones or tens, writing numbers from 0 to 20 is a rote process. Initially, students copy the written numerals while assigning it a name. Over time, children create the understanding that number symbols signify the meaning of counting. Practice count words and written numerals paired with pictures, representations of objects, and objects that represent quantities within the context of life experiences for kindergarteners.

For example, dot cards, dominoes and number cubes all create different mental images for relating quantity to number words and numerals.

One way students can learn the left to right orientation of numbers is to use a finger to write numbers in the air (sky writing). Children will see mathematics as something that is alive and that they are involved in making sense of it.

Since the teen numbers are not written as they are said, teaching the teen numbers as one group of ten and extra ones is foundational to understanding both the concept and the symbol that represents each teen number. For example, when focusing on the number “14,” students could count out fourteen objects using one-to-one correspondence and then use those objects to make one group of ten and four extra ones. Students should connect the representation to the symbol “14” and say, *“Ten and four”.*

Instructional Strategies:

One way students can learn the left to right orientation of numbers is to use a finger to write numbers in the air (sky writing).

Students should study and write numbers 0 to 20 in this order: numbers 1 to 9, the number 0, then numbers 10 to 20.

They need to know that 0 is the number of items left after all items in a set are taken away. Do not accept “none” as the answer to “How many items are left?” for this situation. You want students to answer “zero.”

Resources/Tools

Illustrative Mathematics tasks:

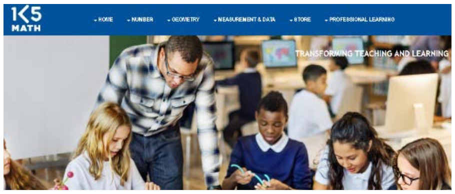
- K.CC Rainbow Number Line
- K.CC Race to the Top
- K.CC Number TIC TAC TOE
- K.CC Assessing Writing Numbers
- K.CC,OA Dice Addition 1
- K.CC Bags of Stuff

NCTM, Focus on Kindergarten.

See Also: K.CC.1 and K.CC.2.

Visit Dr. Doug Clements’ and Dr. Julie Sarama’s website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

Visit **K-5 Math Teaching Resources** section showing the counting stages with corresponding games and activities).



Common Misconceptions:

K.CC.3 addresses the reading and writing of numbers (0-20). Educators need to recognize varied development of fine motor and visual development (such as a reversal of numerals) will occur in a large number of the students. While reversals should be pointed out to students, the emphasis is on the use of numerals to represent quantities rather than the correct handwriting formation of the actual numeral itself. Practice will usually alleviate this problem.

Some students might not see *zero* as a number. Ask students to write 0 and say *zero* to represent the number of items left when all items have been taken away. Avoid using the word *none* to represent this situation.

Domain: Counting and Cardinality (CC)

▶ Cluster B: Count and tell the number of objects.

Standard: K.CC.4

Understand the relationship between numbers and quantities; connect counting to cardinality. **(K.CC.4)**

K.CC.4a.	When counting objects, say each number's name in sequential order, pairing each object with one and only one number name and each number name with one and only one object (Click here for a video showing this concept). **(K.CC.4a)**
K.CC.4b.	Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted. **(K.CC.4b)**
K.CC.4c.	Understand that each successive number name refers to a quantity that is one larger. **(K.CC.4c)**
K.CC.4d.	Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects). **(K.CC.4d)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to other clusters in the Counting and Cardinality Domain and to *classify objects and count the number of objects in each category* in Kindergarten, and to *Add and subtract within 20* in Grade 1.

Explanation and Examples:

This standard asks students to count a set of objects and see sets and numerals in relationship to one another, rather than as isolated numbers or sets. These are higher-level skills that require students to analyze, reason about, and explain relationships between numbers and sets of objects. This standard should first be addressed using numbers 1-5 and building to the numbers 1-10 later in the year. The expectation is that students are comfortable with these skills with numbers 1-10 by the end of Kindergarten.

K.CC.4a reflects the ideas that students implement correct counting procedures by pointing to one object at a time (one-to-one correspondence) using one counting word for each object (one-to-one touching/synchrony), while keeping track of objects that have and have not been counted. This is the foundation of object counting.

K.CC.4b calls for students to answer the question "*How many*?" by counting objects in a set and understanding that the last number stated when counting a set (…8, 9, **10**) represents the total amount of objects: "*There are **10** bears in this pile*." (*cardinality*).

This skill also requires students to understand that the same set counted three different times will end up being the same amount each time. The idea is to develop a purpose for counting while the skill of keeping track of objects is developed. Therefore, a student who moves each object as it is counted recognizes that there is a need to keep track in order to figure out the amount of objects present.

Conservation of number, (regardless of the arrangement of objects, the quantity remains the same); conservation of number is a developmental milestone which some Kindergarten children will not have achieved. The goal of this objective is for students to be able to count a set of objects; regardless of the formation those objects are placed. Teachers will want to ask students if the number of objects will be the same if the count is started at a different spot.

K.CC.4c represents the concept of "one more" while counting a set of objects. Students need to understand that if a set of objects was increased by one then the number name for that set is to be increased by one as well.

Students are asked to understand this concept with and without objects. For example, after counting a set of 8 objects, students should be able to answer the question, "*How many would there be if we added one more object?*"; and answer a similar question when not using objects, by asking hypothetically, "*What if we have 5 cubes and added one more? How many cubes would there be then?*"

This concept should be first taught with numbers 1-5 before building to numbers 1-10. Students should be expected to be comfortable with this skill up to 10 by the end of Kindergarten.

K.CC.4d asks for students to represent a set of objects with a written numeral. The number of objects being recorded should not be greater than 20. Students can record the quantity of a set by selecting a number card/tile (numeral recognition) or writing the numeral. Students can also create a set of objects based on the numeral presented.

Students should be given multiple opportunities to count objects and recognize that a number represents a specific quantity. Once this is established, students begin to read and write numerals (numerals are the symbols for the quantities). The emphasis should first be on quantity and then connecting quantities to the written symbols.

- A sample unit sequence might include:
    - Counting up to 20 objects in many settings and situations over several weeks.
    - Beginning to recognize, identify, and read the written numerals, and match the numerals to given sets of objects.
    - Writing the numerals to represent counted objects.

Instructional Strategies:

One of the first major concepts in a student's mathematical development is cardinality. **Cardinality** is knowing that the number you end on when counting represents the entire amount counted. The big idea is that a number represents an amount and, no matter how you arrange and rearrange the items, the amount is the same. Until this concept is developed, counting is completed without meaning. To determine if students have the cardinality rule, listen to their responses when you discuss counting tasks with them. For example, ask, "*How many are here?*" The student counts correctly and says that there are seven. Then ask, "*Show me seven.*" Students may re-count or will show you the last object counted if they have not developed cardinality. If students show you the last item counted then they believe the numbers are just naming each object; so seven is just the name of the last object.

If students are naming objects with the numbers, make sure you are explicit in showing that when you count to a number that it includes ALL of the objects that were just counted to make that number.

Students with cardinality may emphasize the last count or explain that there are seven because they counted them. These students can now use counting to find a matching set. Students develop the understanding of counting and cardinality from experience. Almost any activity or game that engages children in counting and comparing quantities, such as board games (Candyland, High Ho Cherry O, Chutes and Ladders, Sorry, etc.), will encourage the development of cardinality.

Frequent opportunities to use and discuss counting as a means of solving problems relevant to kindergarteners is more beneficial than repeating the same routine day after day. For example, ask students questions that can be answered by counting up to 20 items as they change locations throughout the school building.

As students develop meaning for numerals, they also compare numerals to the quantities they represent. The models that can represent numbers, such as dot cards and dominoes, become tools for such comparisons. Students can concretely, pictorially or mentally look for similarities and differences in the representations of numbers. They begin to "see" the relationship of one more, one less, two more and two less, thus landing on the concept that successive numbers name quantities that are one larger. In order to encourage this idea, children need discussion and reflection of pairs of numbers from 1 to 10. Activities that utilize anchors of 5 and 10 are helpful in securing understanding of the relationships between numbers. This flexibility with numbers will build students' ability to break numbers into parts.

Provide a variety of experiences in which students connect count words or number words to the numerals that represent the quantities. Students will arrive at an understanding of a number when they acquire cardinality and can connect a number with the numerals and the number word for the quantity they all represent.

Accuracy in counting depends on three things:
1. Knowing the patterns in the number-word so that a correct number-word can be said.
2. Correctly assigning one number-word to one object (one-to one-correspondence).
3. Keeping track of which objects have already been counted so that they are not counted more than once.

Keeping track—differentiating *counted* from *uncounted* objects or entities—is more easily done by moving objects into a counted set. Doing this is not possible with things that cannot be moved, such as pictures in a book. Strategies for keeping track of messy, large sets will continue to be developed for many years.

Regularity and rhythm are important aspects of counting. Activities that increase these aspects can be helpful to children making correspondence errors.

Errors in Counting
Four factors strongly affect accuracy in counting correspondence:
1. Amount of counting experiences (more experience leads to fewer errors)
2. Size of set (children become accurate on small sets first)
3. Arrangements of objects (objects in rows make it easier to keep track of what has been counted and what has not)
4. Effort

From NCTM, Focus in Kindergarten

Instructional Strategies:

Counting should be reinforced throughout the day, not in isolation. (Meaningful Counting) *See Table 6 in Appendix, Developing Counting*

Examples:

- Count the number of empty chairs of the students who are absent.
- Count the number of stairs, shoes, etc.
- Counting groups of ten such as “fingers in the classroom”.
- Count the number of students in various groups at recess.
- Count the number of specific objects they have in their desk (e.g. crayons, pencils, etc.).

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- “How Many Letters in Your Name”, from the Illuminations website. Students review numbers 1 to 10 by counting the number of letters in their names and their classmates' names. They also write and order numbers. The class compiles students' finished product in a class.
- “Let’s Count to 10” is a unit on the Illuminations website in which students make groups of zero to 10 objects, connect number names to the groups, compose and decompose numbers, and use numerals to record the size of a group. Visual, auditory, and kinesthetic activities are included in each lesson. The unit is most appropriate for students learning at the kindergarten and grade one level.

Several studies recommend students count in the manner pictured below. This shows the growing pattern that mimics the number line or the number path. Using several methods for counting creates flexibility in number sense for most students.

Resources/Tools:

Illustrative Mathematics tasks:

- K.CC Goody Bags
- K.CC Counting Mat
- K.CC The Napping House
- K.CC Counting Cup
- K.CC More and Less Handfuls
- K.CC Counting Overview
- K.CC Number Rods
- K.CC Color Week

Georgia Standards Task

See: “Counting Sheep”, NCSM, Great Tasks for Mathematics K-5, (2013).

See Also: K.CC.1

Visit Dr. Doug Clements’ and Dr. Julie Sarama’s website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

LT²

Learning and Teaching with *Learning Trajectories* (LT²)

Visit **K-5 Math Teaching Resources** section showing the counting stages with corresponding games and activities.

Common Misconceptions:

When counting objects student may think the last object counted represents the quantity last “said” When seven chips are lined up and student counts (one-to-one) until s/he gets to seven, when asked to “show” the quantity 7, the student picks up the last chip.

Students may think that the count word used to tag an item is permanently connected to that item. So when the item is used again for counting and should be tagged with a different count word, the student uses the original count word. For example, a student counts four geometric figures: triangle, square, circle and rectangle with the count words: one, two, three, four. If these items are rearranged as rectangle, triangle, circle and square and counted, the student says these count words: four, one, three, two.

Students count objects without seeing sets and numerals in relationship to one another. They see a set of objects as isolated numbers or sets.

Counting on or counting from a given number conflicts with the learned strategy of counting from the beginning. In order to be successful in counting on, students must understand **cardinality** (*the number that ends the counting sequence represents how many objects are in the collection*).

Students often merge or separate two groups of objects and then re-count from the beginning to determine the final number of objects represented. For these students, counting is still a rote skill, or the benefits of counting on have not been realized.

Domain: Counting and Cardinality (CC)

▶ **Cluster B:** Counts and tells the number of objects.

Standard: K.CC.5

Count to answer “how many?” up to 20 concrete or pictorial objects arranged in a line, a rectangular array, or a circle, or as many as 10 objects in a scattered configuration **(subitizing**); given a number from 1 to 20, count out that many objects. **(K.CC.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

See K.CC.4

Explanation and Examples:

This standard addresses counting strategies. From the research in early childhood mathematics, students go through a progression of four general ways to count (Kathy Richardson).

These counting strategies progress from least difficult to most difficult:

- students move objects and count them as they move them,
- students line up the objects and count them,
- students have a scattered arrangement and they touch each object as they count
- students have a scattered arrangement and count them by visually scanning without touching them.

Since the scattered arrangements are the most challenging for students, K.CC.5 calls for students to only count 10 objects in a scattered arrangement, and count up to 20 objects in a line, rectangular array, or circle. Out of these 3 representations, a line is the easiest type of arrangement to count.

Subitizing is specifically identified in this standard. It is not to be confused with one-to-one counting. Subitizing means to “instantly see how many” which refers to the fact that students are identifying quantities but not by physically touching each object. Dr. Doug Clements wrote an article that explains subitizing and how important it is for young learners. Access the article through this link provided by NCTM. Click on the link and then click on “Download PDF” to get the full article.

In one-to-one counting, students should develop strategies to help them organize the counting process to avoid re-counting or skipping objects. Keeping track of what has been counted is a necessary skill.

One of the first major concepts in a student’s mathematical development is **cardinality**. Cardinality, knowing that the number word said tells the quantity and that the number said last represents the entire amount counted.

The big idea is that number means an amount (or a quantity) and, no matter how you arrange and rearrange the items, the amount is the same. Until this concept is developed, counting is merely a routine procedure done when a number is needed. To determine if students have the **cardinality** rule, listen to their responses when you discuss counting tasks with them. For example, ask, "How many are here?" The student counts correctly and says that there are seven. Then ask, "Are there seven?" Students may count or hesitate if they have not developed cardinality. Students with cardinality may emphasize the last count or explain that there are seven because they counted them.

Students can concretely, pictorially, or mentally look for similarities and differences in the representations of numbers. They begin to "see" the relationship of one more, one less, two more and two less, thus landing on the concept that successive numbers name quantities that are one larger (**Hierarchical inclusion**).

Instructional Strategies:

In one-to-one counting, students should develop strategies to help them organize the counting process to avoid re-counting or skipping objects.

Suggestions:

- If items are placed in a circle, the student may mark or identify the starting object.
- If items are in a scattered configuration, the student may move the objects into an organized pattern.
- Some students may choose to use grouping strategies such as placing objects in twos, fives, or tens (note: this is not a kindergarten expectation).
- Counting up to 20 objects should be reinforced when collecting data to create charts and graphs.

Students develop the understanding of counting and cardinality from experience. Almost any activity or game that engages children in counting and comparing quantities, such as board games, will encourage the development of cardinality.

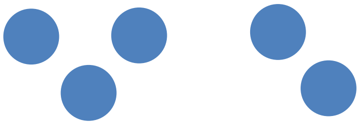
Students should have frequent opportunities to use and discuss counting as a means of solving problems relevant to kindergarteners. This is more beneficial than repeating the same routine day after day. For example, ask students questions that can be answered by counting items up to 20 at various times though out the day (before lunch, after recess, going to PE, etc.).

As students develop meaning for numerals, they should use the numerals to represent quantities. The models, such as dot cards and dominoes that represent numbers, become tools which allow for easier comparisons and conversations. In order to encourage this idea, children need discussion and reflection of pairs of numbers from 1 to 10. Activities that utilize anchors of 5 and 10 are helpful in securing understanding of the relationships between numbers. This flexibility with numbers will build students' ability to break numbers into parts and to think about "friendly numbers."

Provide a variety of experiences in which students connect count words or number words to the numerals that represent the quantities. Read counting books and then provide students bags of items that are related but can be grouped in various ways. Let students determine how they will sort and count the items in order to share with the class.

Subitizing is a skill that most students come to school being able to perceptual subitize up to 3 items. Students do not need to always count one-to-one. Once students begin to subitize quantities, **DO NOT** have them count one-to-one to "check their answer." If students are correct, then acknowledge that they were correct and then ask them how they knew that they were correct. Students should begin to explain that they "saw" a group of 3 and a group of 2 so they knew the total was 5. See example below:

Students are shown the set of dots long enough to see the pattern but not long enough to count one-by-one:



Students tell the total seen and then explain how they knew the answer. Some will say I saw 2 and 1 and 2. This is fine but you want to always ask students to see the largest groupings possible. This supports decomposition of number and understanding the properties of operations in later grades. Also, some research has shown that students who are not able to subitize struggle with knowing their facts and using computational strategies fluently. Dice can be a great tool to reinforce the skill of subitizing by encouraging the students to subitize when they see the dot patterns on the dice.

Resources/Tools

See Also: K.CC.1

Illustrative Mathematics tasks:

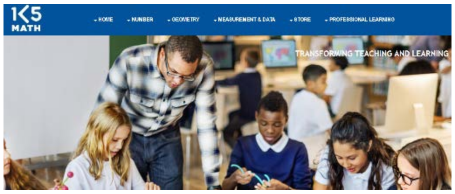
- K.CC Finding Equal Groups
- K.CC Biggest Number Wins
- K.CC More and Less Handfuls

Visit Dr. Doug Clements' and Dr. Julie Sarama's website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

LT²

Learning and Teaching with *Learning Trajectories* (LT²)

Visit **K-5 Math Teaching Resources** section showing the counting stages with corresponding games and activities.



Common Misconceptions:

Some students might think that the count word used to tag an item is permanently connected to that item. In other words, they are "naming" each object that number. For example, a student counts four geometric figures (such as; triangle, square, circle and rectangle) with the count words of one, two, three, four. When asked to show "4 shapes" the student will hold up the rectangle and not all of the shapes. The rectangle is "four" and not all of the shapes together. The student is "naming" or "tagging" the items and not really counting them.

It is suggested that when you are counting with the students that you gather all the items once you have counted and say "This is ___." For example: If you are counting with your fingers to represent 3, hold up one finger for each count and then once you get to three, take your other hand and "group" your three fingers together to show that all three fingers make 3.

1

2

3

This is 3.

Several studies recommend students count in the manner pictured above. This shows the growing pattern that mimics the number line or the number path. Students should understand cardinality of the quantity no matter which method of counting is used.

Domain: Counting and Cardinality (CC)

▶ Cluster C: Compare numbers

Standard: K.CC.6

Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, *(e.g. by using matching and counting strategies).* Include groups with up to ten objects. **(K.CC.6)**

Suggested Standards for Mathematical Practice (MP):

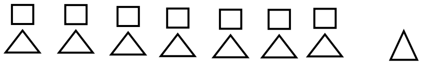
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

See K.CC. 1-5

Explanation and Examples:

This standard expects mastery of up to ten objects. Students can use matching strategies (see Student 1 below), counting strategies or equal shares (see Student 3) to determine whether one group **is greater than**, **less than**, or **equal to** the number of objects in another group (see Student 2).

Student 1	**Student 2**	**Student 3**
I lined up one square and one triangle. Since there is one extra triangle, there are more triangles than squares. 	I counted the squares and I got 8. Then, I counted the triangles and got 9. Since 9 is greater than 8, there are more triangles than squares.	I put them in a pile. I then took away objects. Every time I took a square, I also took a triangle. When I had taken almost all of the shapes away, there was still a triangle left. That means that there are more triangles than squares.

As children develop meaning for numerals, they also compare these numerals to the quantities represented and their number words. Modeling numbers with manipulatives such as dot cards and five- and ten-frames are tools for such comparisons. Children can look for similarities and differences in these different representations of numbers. They begin to "see" the relationship of one more, one less, two more and two less, leading to the concept that successive numbers name quantities that is one larger than the previous number. In order to encourage this idea, children need discussion and reflection of pairs of numbers from 1 to 10.

Children demonstrate their understanding of the meaning of numbers when they can justify why their answer represents a quantity just counted. This justification could merely be the expression that the number said is the total because it was just counted, or a "proof" by demonstrating a one to-one match, by counting again, or other similar means (concretely or pictorially) that makes sense.

An ultimate level of understanding is reached when children can compare two numbers from 1 to10 represented as written numerals without counting.

Students should develop a strong sense of the relationship between quantities and numerals before they begin comparing numbers.

Students state whether the number of objects in a set is more, less, or equal to a set that has 0, 5, or 10 objects.

Instructional Strategies:

In the NCTM publication, Developing Essential Understanding of Number and Numeration, the authors explain that before students can compare or explain greater than or less than, they need to understand if quantities are equal or not equal. First ask if the quantities are equal or not equal and if the answer is not equal, then question further as to "how they are not equal." Build the understanding of more or less after establishing the understanding of equal.

Activities that utilize anchors of **5** and **10** are helpful in securing understanding of the relationships between numbers. (Five-Frames and Ten-Frames are excellent tools!) Flexibility with numbers will greatly impact children's ability to break numbers into parts which is essential in developing understanding of the properties of operations.

Students need to explain their reasoning when they determine whether a number is greater than, less than, or equal to another number. Teachers need to ask probing questions such as "How do you know?" to elicit their thinking.

Strategies:

- Matching: Students use one-to-one correspondence, repeatedly matching one object from one set with one object from the other set to determine which set has more objects.
- Counting: Students count the objects in each set, and then identify which set has more, less, or an equal number of objects.
- Observation: Students may use observation to compare two quantities (e.g., by looking at two sets of objects, they may be able to tell which set has more or less without counting).
- Observations in comparing two quantities can be accomplished through daily routines of collecting and organizing data in displays. Students create object graphs and pictographs using data relevant to their lives (e.g., favorite ice cream, eye color, pets, etc.). Graphs may be constructed by groups of students as well as by individual students.
- Benchmark Numbers: This would be the appropriate time to introduce the use of 0, 5 and 10 as benchmark numbers to help students further develop their sense of quantity as well as their ability to compare numbers.

See also K.CC.4

Illustrative Mathematics tasks:

- K.CC More and Less Handfuls

Resources/Tools:

K.CC.C.6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.*

- Which number is greater? Which number is less? How do you know?

Also See: "It's All in the Bag" Georgia Department of Education. Students will work in partner groups to compare two sets of colored blocks. Discussions should include terms more than, less than, and equal. Students will use counting strategies for sets that have been put together, removed, or are compared.

Visit Dr. Doug Clements' and Dr. Julie Sarama's website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

Visit **K-5 Math Teaching Resources** *Number* section and select *Kindergarten*. This website gives access to several free games and activities for you to use immediately in your classroom.

Common Misconceptions:

One misconception is the relationship between sets and the numbers. Children who are not *conservers* of number will experience difficulties. Also, it is easier for students to identify differences than to find similarities.

Domain: Counting and Cardinality (CC)

▶ Cluster C: Compare numbers

Standard: K.CC.7

Compare two numbers between 1 and 10 presented as written **numerals**. **(K.CC.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to *Work with numbers 11-19 to gain foundations for place value* in Kindergarten, and to all clusters in the Operations and Algebraic Thinking Domain in Grade 1.

Explanation and Examples:

This standard asks students to apply their understanding of numerals 1-10 in order to compare them. For example, looking at the numerals 8 and 10, a student must be able to recognize that the numeral 10 represents a larger quantity than the quantity represented by the numeral 8. Students should begin this standard by having ample experiences with sets of objects (K.CC.3 and K.CC.6) before completing this standard with just numerals. Based on early childhood research, students should not be expected to be comfortable with this skill until the end of Kindergarten.

As children develop meaning for numerals, they also compare these numerals to the quantities and their number words.

Children demonstrate their understanding of the meaning of numbers when they can justify why their answer represents a quantity just counted. This justification could merely be the expression that the number said is the total because it was just counted, or a “proof” by demonstrating a one to-one match, by counting again or other similar means (concretely or pictorially) that makes sense. An ultimate level of understanding is reached when children can compare two numbers from 1 to10 represented as written numerals without counting.

Instructional Strategies:

Modeling numbers with manipulatives such as dot cards and five- and ten-frames are tools for such comparisons. Children can look for similarities and differences in these different representations of numbers. They begin to “see” the relationship of one more, one less, two more and two less, thus landing on the concept that successive numbers name quantities that is one larger than the previous number. In order to encourage this idea, children need discussion and reflection of pairs of numbers from 1 to 10.

Activities that utilize anchors of **5** and **10** are helpful in securing understanding of the relationships between numbers. This flexibility with numbers will impact children’s ability to break numbers into parts which is essential for building an understanding of the properties of operations in future grades.

Resources/Tools

Illustrative Mathematics tasks:

- K.CC Guess the Marbles in the Bag

See Also: K.CC.1

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Visit **K-5 Math Teaching Resources** *Number* section and select *Kindergarten*. This website gives access to several free games and activities for you to use immediately in your classroom.

Common Misconceptions:

See K.CC.6

Domain: Operations and Algebraic Thinking (OA)

▶ **Cluster A:** <u>Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from</u>

Standard: K.OA.1

Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (*e.g. claps*), acting out situations, verbal explanations, expressions, or equations. **(K.OA.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.

Connections:

This cluster is connected to the first OA cluster in first grade (Represent and solve problems involving addition and subtraction).

Explanation and Examples:

All standards in the Operations and Algebraic Thinking (OA) cluster should only include numbers through 10. Students will model simple joining and separating situations with sets of objects, or eventually with equations such as:

5 + 2 = 7 and 7 – 2 = 5

Kindergarten students should see addition and subtraction equations. Student writing of equations in kindergarten <u>is encouraged, but it is not required</u>.

Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

This standard expects students to solve problems by representing addition and subtraction situations in various ways. This objective is primarily focused on <u>understanding</u> the concept of addition and subtraction, rather than merely reading and solving addition and subtraction number sentences (equations) without understanding.

Instructional Strategies: K.OA.1-5

Using addition and subtraction in a word problem context allows students to develop their understanding of what it means to add and subtract.

Students should use objects, fingers, mental images, drawing, sounds, acting out situations and verbal explanations (such as subitizing) in order to develop the concepts of addition and subtraction. Then, they should be introduced to writing expressions and equations using appropriate terminology and symbols which include: **+**, **–**, and **=.**
Assist students in understanding that **+** and **–** are action symbols (operational symbols) at this age, but don't tell them that it will **always** mean addition or subtraction. (These will eventually be used to represent positive and negative.)

- Addition terminology: *add, join, put together, plus, combine, total*
- Subtraction terminology: *minus, take away, separate, difference, compare*

The = symbol doesn't represent an action. This symbol (relational symbol) shows that both sides have the same value (it shows balance).

Have students decompose numbers less than or equal to 5 frequently using a variety of experiences to promote their fluency with sums and differences less than or equal to 5. For example, ask students to use different models to decompose 5 and record their work with drawings or equations.

Next, have students decompose 6, 7, 8, 9, and 10 in a similar fashion. As students begin to understand the role and meaning of arithmetic operations in number systems, they will gain computational fluency, and using efficient and accurate methods for computing.

The teacher can use back-mapping and scaffolding to teach students who show a need for more help with counting. For instance, ask students to build a tower of 5 using 2 green and 3 blue linking cubes while you discuss composing and decomposing 5. Have them identify and compare other ways to make a tower of 5. Repeat the activity for towers of 7 and 9. Help students use counting as they explore ways to compose 7 and 9.

If students progress immediately from working with manipulatives to writing numerical expressions and equations, (skipping the step of using pictorial thinking), students will then be more likely to use finger counting and rote memorization for work with addition and subtraction.
Teachers need to provide instructional experiences so that students will progress from the concrete level, to the pictorial level, then to the abstract level when learning mathematical concepts. (**C**oncrete, **P**ictoria**l**, **A**bstract - **CPA**)

Just knowing the basic facts is not enough. We need to help students develop the ability to quickly and accurately understand the **relationships** between numbers. They need to make sense of numbers as they find and make strategies for joining and separating quantities. (Table 1 in Appendix)

Tools/Resources

See Table 1 in Appendix

For detailed information, see: Learning Progression Operations and Algebraic Thinking

Illustrative Mathematics tasks:

- K.OA Dice Addition 2
- K.OA Ten Frame Addition

Georgia Department of Education:

- "It's All In the Bag" Students will work in partner groups to compare two sets of colored blocks. Discussions should include terms more than, less than, and equal. Students will use counting strategies for sets that have been put together, removed, or are compared.

Visit Dr. Doug Clements' and Dr. Julie Sarama's website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

Visit **K-5 Math Teaching Resources** *Number* section and select *Kindergarten*. This website gives access to several free games and activities for you to use immediately in your classroom.

Common Misconceptions:

Students may over-generalize the vocabulary in word problems and think that certain words indicate solution strategies that must be used to find an answer. They might think that the word *more* always means to add and the words *take away* or *left* always means to subtract. This is not always true. If it is helpful, it is generally helpful for only one-step word problems. Students are expected to solve two-step word problems by 2nd grade and so “key words” actually become a hindrance in understanding and solving problems.

For example – read this problem: *Melisa took 3 stickers she no longer wanted and gave them to Anna. Now Melisa has 5 stickers left. How many stickers did Melisa have to begin with?* This problem is a Take Away-Start Unknown problem (See Common Addition and Subtraction Situations in Appendix). It would be represented with the equation **? – 3 = 5** (situation equation), but the equation that is most helpful to solve the problem is **3 + 5 = ?** (solution equation). If students only look for key words then they would most likely think that 5 – 3 = ? is the equation for this problem, since it is a take away situation. Helping students develop meaning for the operations instead of key words will lead to better foundational understanding.

Note on vocabulary: The term “total” could be used instead of the term “sum”. “Sum” sounds the same as “some”, but has the opposite meaning. “Some” is used to describe problem situations with one or both addends unknown, so it might be more beneficial in the earlier grades to use “total” first and then transition to “sum”.

Formal vocabulary for subtraction (“minuend” and “subtrahend”) is not needed in Kindergarten.

Domain: Operations and Algebraic Thinking (OA)

▶ **Cluster A:** Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from

Standard: K.OA.2

Solve addition and subtraction word problems, and add and subtract within 10, (*e.g. by using objects or drawings to represent the problem.)* Refer to shaded section of Table 1 for specific situation types. **(K.OA.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections:

See Grade K.OA.1

Explanation and Examples:

All standards in the Operations and Algebraic Thinking cluster should only include numbers through 10. Students will model simple joining and separating situations with sets of objects, or eventually with equations such as 5 + 2 = 7 and 7 – 2 = 5.

Kindergarten students should see addition and subtraction equations. Student writing of equations in kindergarten is encouraged, but it is not required. Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

This standard asks students to solve problems presented in a story format (context) with a specific emphasis on using objects or drawings to determine the solution. This builds upon the students understanding of addition and subtraction from K.OA.1 to solve problems. Once again, numbers should not exceed 10.

Teachers should focus on three types of problems during instruction (See Table 1 in Appendix). These are:
Result Unknown, Change Unknown, and Start Unknown. These three types of problems become increasingly difficult for students. Research has found that *Result Unknown* problems are easier than *Change* and *Start Unknown* problems. Kindergarten students should have experiences with all three types of problems but only need to master the ones shaded in Table 1. The level of difficulty can be decreased by using smaller numbers (up to 5) or increased by using larger numbers (up to 10).

Create situations in which students experience the following addition and subtraction problem types (see Table 1, Appendix).

- *Add To* word problems, such as, “Mia had 3 apples. Her friend gave her 2 more. How many does she have now?”
    - A student’s “think aloud” of this problem might be, “I know that Mia has some apples and she’s getting some more. So she’s going to end up with more apples than she started with.”

- *Take From* problems such as:
    - José had 8 markers and he gave 2 away. How many does he have now? When modeled, a student would begin with 8 objects and remove two to get the result.

- *Put Together/Take Apart* problems with Total Unknown gives students opportunities to work with addition in another context such as:
    - There are 2 red apples on the counter and 3 green apples on the counter. How many apples are on the counter?

- Solving *Put Together/Take Apart* problems with Both Addends Unknown provides students with experiences with finding all the decompositions of a number and investigating the patterns involved.
    - There are 10 apples on the counter. Some are red and some are green. How many apples could be green? How many apples could be red?

Using a word problem context allows students to develop their understanding about what it means to add and subtract. (*Addition is putting together and adding to. Subtraction is taking apart and taking from)*. Instruction that helps Kindergarteners to develop the concept of addition/subtraction is modeling the actions in word problem using objects, fingers, mental images, drawings, sounds, acting out situations, and/or verbal explanations.

Students may use different representations based on their experiences, preferences, etc. They may connect their conceptual representations of the situation using symbols, expressions, and/or equations. Students should experience the addition and subtraction problem types found in Table 1, Appendix).

Instructional Strategies:
See K.OA.1

Tools/Resources
See: K-5 Number and Operations in Base Ten Learning Progression for additional detailed information.

Illustrative Mathematics tasks:

- K.OA What's Missing?
- K.CC,OA Dice Addition 1
- K.OA Dice Addition 2
- K.OA Ten Flashing Fireflies

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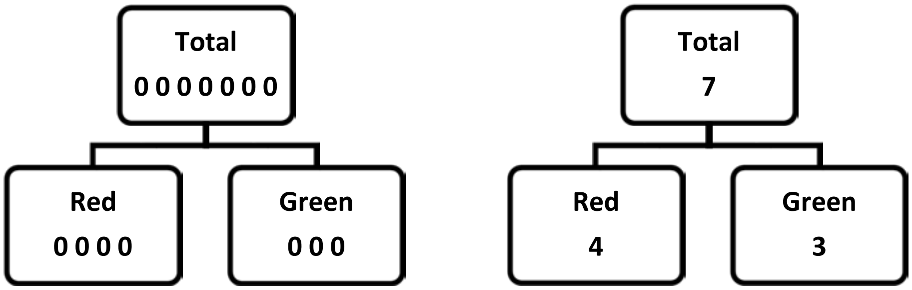
Common Misconceptions:
See K.OA.1

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster A:*** *Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from*

Standard: K.OA.3

Decompose numbers less than or equal to 10 into pairs in more than one way, (*e.g. by using objects or drawings, and record each decomposition by a drawing or equation* (*e.g.* $5 = 2 + 3$ *and* $5 = 4 + 1$)). **(K.OA.3)**



Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning

Connections:

This is connected to K.CC.7 and 1.OA.3, 1.OA.4, 1.OA.6 and 1.OA.8.

Explanation and Examples:

All standards in the Operations and Algebraic Thinking cluster should only include numbers through 10. Students will model simple joining and separating situations with sets of objects, or eventually with equations such as 5 + 2 = 7 and 7 – 2 = 5.

Kindergarten students should see addition and subtraction equations. Student writing of equations in kindergarten is encouraged, but it is not required. Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

This standard asks students to understand that a set of (5) object can be broken into two sets (3 and 2) and still be the same total amount (5). The focus is on number pairs which add to a specified total, 1-10. Students need to have a firm grasp of conservation of number.

In addition, this standard asks students to understand that a set of objects (5) can be broken in multiple ways (3 and 2; 4 and 1). Thus, when breaking apart a set (decomposing), students develop the understanding that a smaller set of objects exists within that larger set (inclusion). This should be developed in context before moving into representing decomposition with symbols (+, -, =).

After the students have had numerous experiences with decomposing sets of objects and recording with pictures and numbers, the teacher eventually makes connections between the drawings and symbols*: 5=4+1, 5=3+2, 5=2+3, and 5=1+4*

Equations come after much work with pictures or manipulatives, and students should never give the equation without a pictorial or concrete representation.

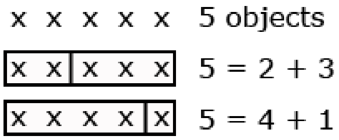
Example:
"Bobby Bear is missing 5 buttons on his jacket. How many ways can you use blue and red buttons to finish his jacket? Draw a picture of all your ideas."

Students could draw pictures of:

4 blue and 1 red buttons
3 blue and 2 red buttons
2 blue and 3 red buttons
1 blue and 4 red buttons

Students may use objects such as cubes, two-color counters, square tiles, etc. to show different number pairs for a given number. For example, for the number 5, students may split a set of 5 objects into 1 and 4, 2 and 3, etc.

After the students have had numerous experiences with decomposing sets of objects and recording with pictures and numbers, the teacher eventually makes connections between the drawings and symbols*: 5=4+1, 5=3+2, 5=2+3, and 5=1+4.*



Instructional Strategies:
See K.A0.1

Sample unit sequence:

- A contextual problem (word problem) is presented to the students such as, "Melisa goes to Debbie's house. Debbie tells her she may have 5 pieces of fruit to take home. There are lots of apples and bananas. How many of each can she take?"
- Students find related number pairs using objects (such as cubes or two-color counters), drawings, and/or equations. Students may use different representations based on their experiences, preferences, etc.
- Students may write equations that equal 5 such as:

* 5=4+1
* 3+2=5
* 2+3=4+1
* 5+ 0 = 5

This is a good opportunity for students to systematically list all the possible number pairs for a given number. For example, all the number pairs for 5 could be listed as 0+5, 1+4, 2+3, 3+2, 4+1, and 5+0. Students should describe the pattern that they see in the addends, e.g., each number is one less or one than the previous addend. (Continue to make sure students include the number plus zero as a possible solution).

Tools/Resources:

Illustrative Mathematics tasks:

- K.OA Make 9
- K.OA Bobbie Bear's Buttons
- K.OA Christina's Candies
- K.OA Pick Two
- K.OA Shake and Spill
- K.OA My Book of Five

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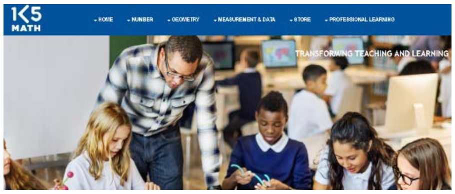
LT²

Learning and Teaching with *Learning Trajectories* (LT²)



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Common Misconceptions:

See K.OA.1

Domain: Operations and Algebraic Thinking (OA)

▶ **Cluster A:** <u>Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from</u>

Standard: K.OA.4

For any number from 1 to 9, find the number that makes 10 when added to the given number, *(e.g. by using objects or drawings, and record the answer with a drawing or equation.).* **(K.OA.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

See K.OA.1

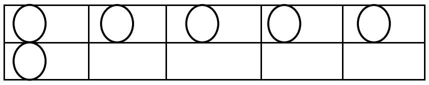
Explanations and Examples:

This standard builds upon the understanding that a number can be decomposed into parts (K.OA.3).

The number pairs that total ten are foundational for students' ability to work fluently within numbers and operations. Different models, such as ten-frames, cubes, two-color counters, etc., assist students in visualizing these number pairs for ten. Once students have had experiences breaking apart ten into various combinations, then ask students to find a *missing* part of 10.

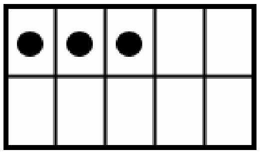
Example 1:

"A full case of juice boxes has 10 boxes. There are only 6 boxes in this case. How many juice boxes are missing?"

Student 1	**Student 2**	**Student 3**
Using a Ten Frame "I used 6 counters for the 6 boxes of juice still in the case. There are 4 blank spaces so 4 boxes have been removed. This makes sense since 6 and 4 more equal 10". 	Think addition. "I counted out 10 cubes because I knew there needed to be ten. I pushed these 6 over here because they were in the container. These are left over. So there's 4 missing."	Basic Fact. "I know that it's 4 because 6 and 4 is the same amount as 10".

Example 2:
Students place three objects on a ten frame and then determine how many more are needed to “make a ten.”

Students may use electronic versions of ten frames to develop this skill.



Example 3:
The student snaps ten cubes together to make a “train.”

- Student breaks the “train” into two parts. S/he counts how many are in each part and record the associated equation (10 = ___ + ___).
- Student breaks the “train into two parts. S/he counts how many are in one part and determines how many are in the other part without directly counting that part. Then s/he records the associated equation (if the counted part has 4 cubes, the equation would be 10 = 4 + ___).
- Student covers up part of the train, without counting the covered part. S/he counts the cubes that are showing and determines how many are covered up. Then s/he records the associated equation (if the counted part has 7 cubes, the equation would be 10 = 7 + ___).

Example 4:
The student tosses ten two-color counters on the table and records how many of each color are facing up.

Instructional Strategies:
See K.OA.1

Tools/Resources:
See EngageNY Modules:

See: K-5 Number and Operations in Base Ten Learning Progression for additional detailed information.

Visit Dr. Doug Clements’ and Dr. Julie Sarama’s website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

Visit **K-5 Math Teaching Resources** *Number* section and select *Kindergarten*. This website gives access to several free games and activities for you to use immediately in your classroom.

Common Misconceptions:
See K.OA.1

Domain: Operations and Algebraic Thinking (OA)

▶ **Cluster A:** Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from

Standard: K.OA.5

Fluently (efficiently, accurately, and flexibly) add and subtract within 5. **(K.OA.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning

Connections:

See K.OA.1

Explanation and Examples:

All standards in the Operations and Algebraic Thinking cluster should only include numbers through 10. Students will model simple joining and separating situations with sets of objects, or eventually with equations such as 5 + 2 = 7 and 7 – 2 = 5.

Kindergarten students should see addition and subtraction equations. Student writing of equations in kindergarten is encouraged, but it is not required. Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

This standard uses the word fluently, which means accuracy (correct answer), efficiency (a reasonable amount of steps), and flexibility (using strategies such as the distributive property and/or those shown below). Fluency is developed by working with many different kinds of objects over an extended amount of time.

This objective does not require students to instantly know the answer. ***Traditional flash cards or timed tests have not been proven as effective instructional strategies for developing fluency. In fact, research by Jo Boaler has found these to be detrimental to long-term retention.***

This standard focuses on students being able to add and subtract numbers within 5. Adding and subtracting fluently refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

Strategies students may use to attain fluency include:

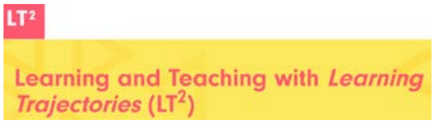
- Counting on (e.g., for 3+2, students will state, “3,” and then count on two more, “4, 5,” and state the solution is “5”)
- Counting back (e.g., for 4-3, students will state, “4,” and then count back three, “3, 2, 1” and state the solution is “1”)
- Counting up to subtract (e.g., for 5-3, students will say, “3,” and then count up until they get to 5, keeping track of how many they counted up, stating that the solution is “2”)
- Using doubles (e.g., for 2+3, students may say, “I know that 2+2 is 4, and 1 more is 5”)
- Using commutative property (e.g., students may say, “I know that 2+1=3, so 1+2=3”)
- Using fact families (e.g., students may say, “I know that 2+3=5, so 5-3=2”)
- Students may use electronic versions of five frames to develop fluency of these facts.

Instructional Strategies:
See K.OA.1

Tools/Resources:
Illustrative Mathematics tasks:

- K.OA Many Ways to Do Addition 1
- K.OA My Book of Five

Visit Dr. Doug Clements’ and Dr. Julie Sarama’s website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.



Visit **K-5 Math Teaching Resources** *Number* section and select *Kindergarten*. This website gives access to several free games and activities for you to use immediately in your classroom.

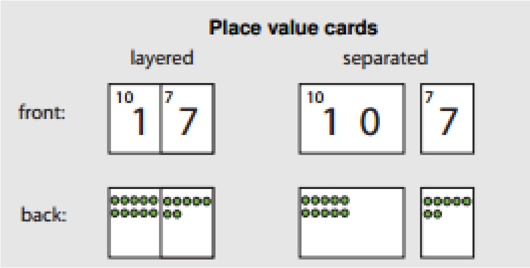
Common Misconceptions:
See K.OA.1

Domain: Number and Operations in Base Ten (NBT)

▶ **Cluster A:** Work with numbers 11-19 to gain foundations for place value.

Standard: K.NBT.1

Compose and **decompose** numbers from 11 to 19 into ten ones and some further ones, *(e.g. by using objects or drawings, and record each composition or decomposition by a drawing or equation* *(e.g.* $10 + 8 = 18 \text{ and } 19 = 10 + 9$*);* *)*; understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones. **(K.NBT.1)**



Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to K.CC.3; and language arts standards: K.RI.3; K.W.2

Explanation and Examples:

This is the first time that students move beyond the number 10 with representations, such as objects (manipulatives) or drawings.

The spirit of this standard is that students separate out a set of 11-19 objects into a group of ten objects with leftovers. This ability is a pre-cursor to later grades when they need to understand a more complex concept that a group of 10 objects is also one ten (*unitizing*). Ample experiences with ten frames will help solidify this concept.

Research states that students are not ready to unitize until the end of first grade. Therefore, this work in Kindergarten lays the foundation of composing tens and recognizing leftovers.

Example:

Teacher: "Please count out 15 chips."

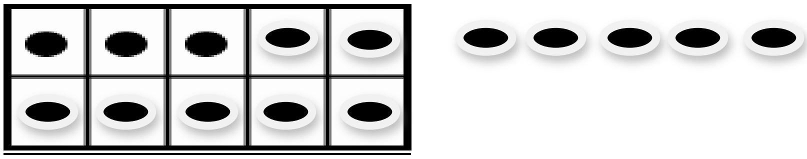
Student: Student counts 15 counters (chips or cubes).

Teacher: "Do you think there is enough to make a group of ten chips? Do you think there might be some chips leftover?"

Student: Student answers.

Teacher: "Use your counters to find out."

Student: Student can either fill a ten frame or make a stick of ten connecting cubes. They answer, "There is enough to make a group of ten"



Teacher: How many leftovers do you have?

Student: Students say, "I have 5 left over."

Teacher: How could we use words and/or numbers to show this?

Student: Students might say "Ten and five is the same amount as 15", "15 = 10 + 5"

Special attention needs to be paid to this set of numbers as they do not follow a consistent pattern in the verbal counting sequence.

- Eleven and twelve are special number words.
- "Teen" means one "ten" plus ones.
- The verbal counting sequence for teen numbers is backwards – we say the ones digit before the tens digit. For example "27" reads tens to ones (twenty-seven), but 17 reads ones to tens (seven-teen).
- In order for students to interpret the meaning of written teen numbers, they should read the number as well as describe the quantity. For example, for 15, the students should read "fifteen" and state that it is one group of ten *and* five ones and record that 15 = 10 + 5.

Teaching the **teen numbers** as one group of ten and extra ones is foundational to understanding both the concept and the symbol that represent each teen number. For example, when focusing on the number "14," students should count out fourteen objects using one-to-one correspondence and then use those objects to make one group of ten ones and four additional ones. Students should connect the representation to the symbol "14." Students should recognize the pattern that exists in the teen numbers; every teen number is written with a 1 (representing one ten) and ends with the digit that is first stated.

Wright (2006) described a progression in children's understanding of ten:

1. Initial concept of ten: understands ten as ten ones but does not see ten as a unit.
2. Intermediate concept of ten: understands ten as a unit composed of ten ones but needs manipulatives or drawings to help solve problems involving tens.
3. Facile concept of ten: can solve tasks involving tens and ones without materials and can mentally think about two-digit numbers as groups of tens and ones.

Instructional Strategies:
Kindergarteners need to understand the idea of *a ten* so they can develop the strategy of adding onto 10 to add within 20 in Grade 1. Students need to construct their own base-ten ideas about quantities and their symbols by connecting to counting by ones. They should use a variety of manipulatives to model and connect equivalent representations for the numbers 11 to19. For instance, to represent 13, students can count by ones and show 13 beans. They can anchor to five and show one group of 5 beans and 8 beans or anchor to ten and show one group of 10 beans and 3 beans.

Students need to eventually see *a ten* as different from 10 ones.

After the students are familiar with counting up to 19 objects by ones, have them explore different ways to group the objects that will make counting easier. Have them estimate before they count and group. Discuss their groupings and lead students to conclude that grouping by ten is desirable. *10 ones make 1 ten* makes students wonder how something that means a lot of things can be one thing.

Students need to first use materials that can be grouped to represent numbers 11 and 19 because a group of ten such as a bundle of 10 straws or a cup of 10 beans makes more sense than *a ten* in tradeable materials. They need to see that there are 10 single objects represented on the item for ten in tradeable materials, such as the rod in base-ten blocks.

Students need to learn to attach words to materials and groups and understand what they represent. Eventually, they need to see the rod as *a ten* that they did not group themselves.

Students should impose their base-ten concepts on a model made from “groupable” and “tradeable” materials. Students can transition from “groupable” to “tradeable” materials by leaving a group of ten intact to be reused as a tradeable item.

When using tradeable materials, students should reflect on the ten-to-one relationships in the materials, such as the “tenness” of the rod in base-ten blocks. After many experiences with tradeable materials, students can use dots and a stick (one tally mark) to record ones and a ten. Kindergartners should use **proportional base-ten models**, where a group often is physically 10 times larger than the model for one. *Non-proportional models* such as an abacus and money should not be used at this grade level.

Encourage students to use base-ten language to describe quantities between 11 and 19. At the beginning, students do not need to use *ones* for the singles. Some of the base-ten language that is acceptable for describing quantities such as18 includes *one ten and eight, a bundle and eight*, *a rod and 8 singles* and *ten and eight more.* Write the horizontal equation 18 = 10 + 8 and connect it to base-ten language.

Encourage, but do not require, students to write equations to represent quantities. Students have difficulty with *ten* as a singular word that means 10 things. For many students, the understanding that a group of 10 things can be replaced by a single object and they both represent 10 is confusing.

Tools/Resources

See: K-5 Number and Operations in Base Ten Learning Progression for additional detailed information.

Illustrative Mathematics tasks:

- K.NBT What Makes a Teen Number?

Visit Dr. Doug Clements' and Dr. Julie Sarama's website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

LT²

Learning and Teaching with *Learning Trajectories* (LT²)

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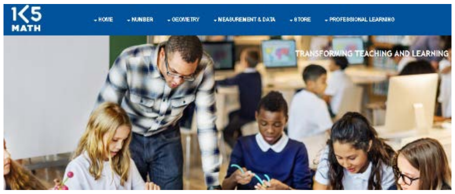

Common Misconceptions:

Students have difficulty with *ten* as a singular word that means 10 things. For many students, the understanding that a group of 10 things can be replaced by a single object and they both represent 10 is confusing. Help students develop the sense of *ten* by first using groupable materials then replacing the group with an object or representing 10, such as a rod or 10 Frame.

Watch for and address the issue of attaching words to materials and groups without knowing what they represent. If this misconception is not addressed early on it can cause additional issues when working with numbers 11-19 and beyond.

At this stage you may encounter some students who when working with "grouped" materials will continue to count each object in the "ten group". The students who do this are developmentally at the beginning of the idea of ten as a group.

Domain: Measurement and Data (MD)

● Cluster A: Describe and compare measurable attributes.

Standard: K.MD.1

Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object. **(K.MD.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

See the Early Childhood Standards.

Explanation and Examples:

This standard calls for students to describe measurable attributes of objects, such as **length** and **weight**, In order to describe these attributes; students must have many opportunities to informally explore them.

- Students should state comparisons of objects verbally and then focus on specific attributes when making verbal comparisons for K.MD.2.
- They may identify measurable attributes such as length, width, height, and weight. For example, when describing a soda can, a student may talk about how tall, how wide, how heavy, or how much liquid can fit inside. These are all measurable attributes. Non-measurable attributes would include: words on the object, colors, pictures, etc.

Recommended sequence for measurement instruction:

1. Making Comparisons: understand the attribute to be measured. Use direct comparisons as much as possible based on the specified attribute – longer, lighter, taller, etc.
2. Using Models of Measuring Units – understand how filling, covering, matching or making other comparisons of an attribute with measuring units produces a number called a **measure**. Use physical models of measuring units by starting with nonstandard units and then moving to measuring tools.
3. Using Measuring instruments – use common measuring tools with understanding and flexibility. Making measuring tools and then using them bridges the understanding to using rulers and other standard measurement tools.

Instructional Strategies:

It is critical for students to be able to identify and describe measureable attributes of objects. An object has different attributes that can be measured, like the height and weight of a can of food.

Students should be given many opportunities to compare directly so the attribute becomes the focus. For example, when comparing the volume of two different boxes, ask students to discuss and justify their answers to these questions: Which box will hold the most? Which box will hold least? Will they hold the same amount? “How could you find out?” Students can decide to fill one box with dried beans then pour the beans into the other box to determine the answers to these questions.

Have students work in pairs to compare their arm spans. As they stand back-to-back with outstretched arms, compare the lengths of their spans, then determine who has the shortest arm span. Ask students to explain their reasoning.

Then ask students to suggest other measureable attributes of their bodies that they could directly compare, such as their height or the length of their feet.

Tools/ Resources

For detailed information, see K-5 Geometric Measurement Learning Progression.

Illustrative Mathematics tasks:

- K.MD Longer and Heavier? Shorter and Heavier?
- K.MD Which is heavier?

Georgia Department of Education:

- “Chrysanthemum is My Name”, Georgia Department of Education. In this task, the students will compare the length of their names with the length of their classmate’s names using cube towers. They will share and discuss the differences and similarities in the lengths using math vocabulary such as longer, shorter, more and less.

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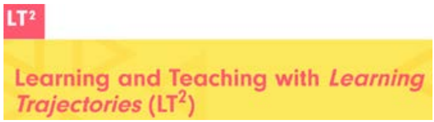
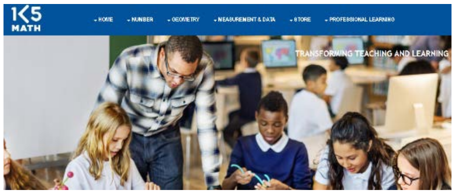


Visit **K-5 Math Teaching Resources** *Measurement & Data* section and select *Kindergarten*. This website gives access to several free games and activities for you to use immediately in your classroom.



Common Misconceptions:

Some students may experience difficulty in being able to classify the same object into different categories. They see it as belonging to only one category. More experiences and conversations will help the students understand how one object can be in multiple categories. This is essential for future work in later grades.

Domain: Measurement and Data (MD)

● **Cluster A:** Describe and compare measurable attributes.

Standard: K.MD.2

Directly compare two objects, with a measureable attribute in common, to see which object has “more of”/”less of” the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.* **(K.MD.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

This is connected to K.MD.1. Also connected to *Measure lengths indirectly and by iterating length units* in Grade 1.

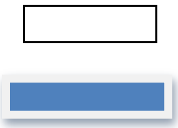
Explanation and Examples:

This standard asks for direct comparisons of objects. Direct comparisons are made when objects are put next to each other, such as two children, two books, two pencils. For example; a student may line up two blocks and say, “This block is a lot longer than this one.” Students are not comparing objects that cannot be moved and lined up next to each other. The objects do not have to be the same. A book might be compared to a pencil or a pencil to a crayon.

When making direct comparisons for length, students must attend to the “starting point” of each object and recognize that objects should be matched up at the end of objects to get accurate measurements. For example, the ends need to be lined up at the same point, or students need to compensate when the starting points are not lined up.

Conservation of length includes understanding that if an object is moved, its length does not change; an important concept when comparing the lengths of two objects. This is a developmental milestone for young children, children need multiple experiences to move beyond the idea that…
*“Sometimes this block is **longer than** this one and sometimes it’s **shorter** (depending on how I lay them side by side) and that’s okay.” “This block is always longer than this block (with each end lined up appropriately).”*

Before conservation of length: *The blue block is longer or shorter than the plain block when they are lined up like this. But when I move the blocks around, sometimes the plain block is longer than the blue block.*



After conservation of length: *I have to line up the blocks to measure them.*

Language plays an important role in this standard as students describe the similarities and differences of measurable attributes of objects (e.g., shorter than, taller than, lighter than, the same as, etc.).

Instructional Strategies:
Students should have many opportunities to compare the lengths of two objects both directly (by comparing them with each other) and indirectly (by comparing both with a third objects.

A student can be given an object as part of a scavenger hunt in the classroom and be asked to find one or two objects that are the same length as; two that are longer and two that are shorter.

Tools/Resources
Illustrative Mathematics tasks:

- K.MD Size Shuffle
- K.MD Which weighs more? Which weighs less?
- K.MD Which is heavier?
- K.MD Longer and Shorter
- K.MD Which is Longer?
- K.MD Which is Heavier?

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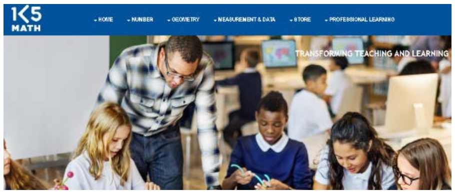

Common Misconceptions:
Many students have difficulty understanding that when an object is moved away from the object they are comparing it with, the length does not change. With multiple opportunities, students learn that they have to line up the items they are comparing and/or measuring. *(Conservation of Length: includes understanding that if an object is moved, its length does not change; an important concept when comparing the lengths of two objects).*

Domain: Measurement and Data (MD)

◆ **Cluster B:** Classify objects and count the number of objects in each category.

Standard: K.MD.3

Classify objects into given categories; count the numbers of objects in each category and sort the categories by count (*Limit category counts to be less than or equal to 10*). **(K.MD.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

See the Early Childhood Standards.

This cluster is connected to *Know number names and the count sequence* and *Count to tell the number of objects* in Kindergarten, and to *Represent and interpret data* in Grade 1.

Explanation and Examples:

This standard asks students to identify similarities and differences between objects (e.g., size, color, shape) and use the identified attributes to **sort** a collection of objects. Once the objects are sorted, the students count the amount in each set, then the students are asked to give category names for each group.

For example; when given a collection of buttons, the student separates the buttons into different piles based on color (all the blue buttons are in one pile, all the orange buttons are in a different pile, etc.). Then the student counts the number of buttons in each pile: blue (5), green (4), orange (3), purple (4). Finally, the student gives each group a name (category).

Other possible objects to sort include: shells, geometric shapes, beans, small toys, coins, rocks, etc. After sorting and counting, it is important for students to:

- *explain how they sorted the objects;*
- *label each set with a category;*
- *answer a variety of counting questions that ask, "How many …"; and*
- *compare sorted groups using words such as, "most", "least", "alike" and "different".*

This objective helps to build a foundation for data collection in future grades. In later grade, students will transfer these skills to creating and analyzing various graphical representations.

Instructional Strategies:

Provide categories for students to use to sort a collection of objects. Each category can relate to only one attribute, like *Red* and *Not Red* or *Hexagon* and *Not Hexagon,* and contain up to 10 objects. Students count how many objects are in each category and then order the categories by the number of objects they contain.

Have students count the objects in each category and order the categories by the number of objects they contain.

You can have students infer the classification of objects by guessing the rule for a sort. First, the teacher uses one attribute to sort objects into two groups or regions without labels. The students are to determine how the objects were sorted, suggesting labels for the two categories and explaining their reasoning.

Attribute materials are great manipulatives to use for these types of activities. Attribute materials can be commercially bought (attribute blocks, attribute bears, attribute people, etc.) or you can prepare your own materials. Here is a sample set.

Tools/Resources

Illustrative Mathematics tasks:

- K.MD Sort and Count I
- K.MD Sort and Count II
- K.MD Goodie Bags

Georgia Department of Education:

- "Take it to the store" for some ideas that could be used.

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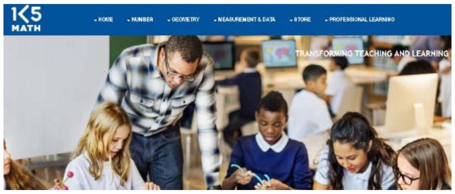

Common Misconceptions:

Some students may have difficulty with placing objects in categories because they are uncertain of the attribute that is being targeted. When using attribute blocks some are thick and some are thin. Discussing the terms for all students, especially your low-language students, will alleviate these difficulties.

Domain: Geometry (G)

● **Cluster A:** Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

Standard: K.G.1

Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above, below, beside, in front of, behind, and next to*. **(K.G.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

See the Early Childhood Standards.

This cluster is connected to *Analyze, compare, create and compose shape*s in Kindergarten, and to *Reason with shapes and their attributes* in Grade 1.

Explanation and Examples:

This standard expects students to use positional words such as: *above*, *below*, *beside*, *in front of*, *behind*, and *next to* in describing objects in the environment.

Kindergarten students need to focus first on location and position of two-and-three-dimensional objects in their classroom prior to describing location and position of two-and-three-dimension representations on paper. Examples of environments in which students would be encouraged to identify shapes would include the classroom and outside on the playground using positional words in their descriptions.

Teachers should work with children and pose four mathematical questions: Which way? How far? Where? And what objects? To answer these questions, children develop a variety of important skills contributing to their spatial thinking.

Examples:

- Teacher holds up an object such as an ice cream cone, a number cube, a basketball, etc. and asks students to identify the shape. Example: The teacher holds up a can of soup and asks,” What shape is this can?” Students respond “cylinder!”
- Teacher places an object *next to, behind, above, below, beside*, or *in front of* another object and asks positional questions. Where is the water bottle? (water bottle is placed *behind* a book) Students say “The water bottle is behind the book.”

Students should have multiple opportunities to identify shapes; these may be displayed as photographs, or pictures using the document camera or interactive whiteboard after considerable work with concrete objects.

Instructional Strategies:

Develop spatial sense by connecting geometric shapes to students’ everyday lives. Initiate natural conversations about shapes in the environment. Have students identify and name two- and three-dimensional shapes inside and outside of the classroom and describe their relative position.

Ask students to find rectangles in the classroom and describe the relative positions of the rectangles they see, e.g. *This rectangle (a poster) is over the sphere (globe)*. Teachers and students can use a digital camera to record these relationships.

Hide shapes around the room. Have students say where they found the shape using positional words, e.g. *I found a triangle UNDER the chair*.

Have students create drawings involving shapes and positional words: *Draw a window ON the door* or *Draw an apple UNDER a tree*. Some students may be able to follow two- or three-step instructions to create their drawings.

Use a shape in different orientations and sizes along with non-examples of the shape so students can learn to focus on defining attributes of the shape.

Manipulatives used for shape identification actually have three dimensions. However, Kindergartners need to think of these shapes as two-dimensional or “flat” and typical three-dimensional shapes as “solid.” Students will identify two-dimensional shapes that form surfaces on three-dimensional objects. Students need to focus on noticing two and three dimensions, not on the words *two-dimensional* and *three-dimensional.*

You may create a game where an object is identified and whispered into the teacher’s ear. Students then ask question of the first student---“Is it in front of ____”, “Does it have the shape of a sphere?”, etc.

Resources/Tools

For detailed information see Geometry Learning Progression:

Illustrative Mathematics tasks:

- K.G Shape Sequence Search
- K.G Shape Hunt Part 1
- K.G Shape Hunt Part 2

Visit Dr. Doug Clements’ and Dr. Julie Sarama’s website, Learning Trajectories, for access to their developmental progressions that give you specific activities and lessons to help move students along in their understanding of mathematics.

Visit **K-5 Math Teaching Resources** *Geometry* section and select *Kindergarten*. This website gives access to several free games and activities for you to use immediately in your classroom.

Common Misconceptions:

Students many times use incorrect terminology when describing shapes. For example, students may say a *cube* is a *square* or that a *sphere* is a *circle*. The use of the two-dimensional shape that appears to be part of a three-dimensional shape to name the three-dimensional shape is a common misconception. Work with students to help them understand that the two-dimensional shape is a part of the object but it has a different name.

Domain: Geometry (G)

● **Cluster A:** Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

Standard: K.G.2

Correctly gives most precise name of shapes regardless of their orientations (position and direction in space) or overall size. **(K.G.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected to *Analyze, compare, create and compose shape*s in Kindergarten, and to *Reason with shapes and their attributes* in Grade 1.

Explanation and Examples:

Children need to see examples of shapes beyond circles, squares, rectangles, and triangles. Without this exposure, children develop limited notions about geometry. Kindergartners should also learn to recognize these shapes whether they are in "standard position" or rotated so that it is not in the same position every time.

Kindergartners can begin to develop explicit and sophisticated levels of thinking and communication. They can learn to describe, and even define, shapes in terms of their *parts* or *attributes* (properties). For example, they can build accurate representations of shapes from physical models of line segments, such as craft sticks. When children discuss what they have built, attributes of the shapes will arise naturally.

Example:

Student: "I built a rectangle"
Teacher: "How do you know it is a rectangle?"
Student: **"**Because the two opposite sides are the same length and all the angles are the same."

The experience of discussing attributes of rectangles (or any shape they build) helps children begin to understand the *geometric structure* of all rectangles at an explicit level of thinking.

Having students build shapes, or to feel shapes hidden in a bag or box and describe what they feel, helps students explore the properties/attributes of shapes. Such activities help children learn to identify and describe shapes by the number of their sides and corners. Such descriptions build geometric concepts, and reasoning skills and language.

You want students to describe shapes in terms of their properties, such as saying that squares have four side of *equal length*. They can informally describe properties of blocks in functional contexts, such as "These blocks roll but those do not roll".

Instructional Strategies:
Allow children to build shapes using straws or toothpicks for the edges and clay or marshmallows as the connectors (vertices).

Another valuable activity is the tactile-kinesthetic exploration of shapes—feeling shapes hidden in a box. Kindergartners can name the shape they are feeling rather than just match shapes. After this, they can extend the activity further as they **describe** the shape without using its name, so that their friends can name the shape. In this way, children learn the properties of the shape, moving from intuitive to explicit, verbalized knowledge. All these variations can be repeated with less familiar shapes.

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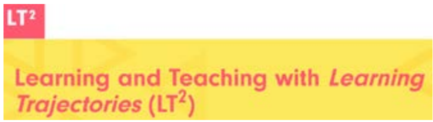

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Common Misconceptions:
Many children come to believe incorrectly that shapes such as a trapezoid "is not a shape" because it is not a shape for which they know a name for it. Explaining other shapes and providing the students the names for the shapes is appropriate if they are curious, but you would not expect mastery of the other shapes that are explored.

Domain: Geometry (G)

● **Cluster A:** Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

Standard: K.G.3
Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid"). **(K.G.3)**

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:
This cluster is connected to *Analyze, compare, create and compose shape*s in Kindergarten, and to *Reason with shapes and their attributes* in Grade 1.

Explanation and Examples:
Students are asked to identify two-dimensional (flat objects) and three-dimensional (solid objects). This standard can be done by having students sort 2-dimensional and 3-dimensional objects, or by having students describe the appearance or thickness of shapes.

Student should be able to differentiate between two dimensional and three dimensional shapes.
- Student names a picture of a shape as two dimensional because it is flat and can be measured in only ***two*** ways (length and width).
- Student names an object as three dimensional because it is not flat (it is a solid object/shape) and can be measured in ***three*** different ways (length, width, height/depth).

It is important to note that faces of three dimensional shapes can be identified as specific two-dimensional shapes.

Instructional Strategies:
See K.G.2

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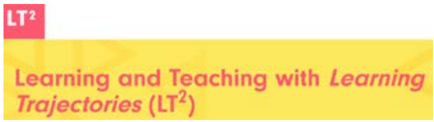

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Common Misconceptions:
See K.G.2

Domain: Geometry (G)

◆ **Cluster B:** Analyze, compare, create, and compose shapes.

Standard: K.G.4

Analyze and compare two- and three-dimensional shapes, in different sizes and orientations (position and direction in space), using informal language to describe their similarities, differences, parts *(e.g. number of sides and vertices/"corners"*) and other attributes *(e.g. having sides of equal length)*. **(K.G.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected *to Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres)* in Kindergarten, and to *Reason with shapes and their attributes* in Grade 1.

Explanation and Examples:

This standard asks the student to note similarities and differences between and among 2-D and 3-D shapes using informal language. These experiences help young students begin to understand how 3-dimensional shapes are composed of 2-dimensional shapes (e.g.., the base and the top of a cylinder is a circle; a circle is formed when tracing a sphere).

Students analyze and compare two- and three-dimensional shapes by observations. Their visual thinking enables them to determine if things are alike or different based on the appearance of the shape.

Students sort objects based on appearance. Even in early explorations of geometric properties, students are introduced to how categories of shapes are subsumed (*contained*) within other categories. For instance, they will recognize that a square is a special type of rectangle.

Students should be exposed to triangles, rectangles, and hexagons whose sides are not all congruent. If students always see regular shapes, they will form misconceptions.

Students will first begin to describe shapes using everyday language and then refine their vocabulary to include sides and vertices/corners.

Instructional Strategies:

Opportunities to work with pictorial representations, concrete objects, as well as technology, helps student develop their understanding about these shapes and to build a varied descriptive vocabulary for both two- and three-dimensional shapes.

Use shapes collected from students to begin the investigation into basic properties and characteristics of two- and three-dimensional shapes. Have students analyze and compare each shape with other objects in the classroom and describe the similarities and differences between the shapes.

Ask students to describe the shapes while the teacher records key descriptive words in common student language. Students may use the word *flat* to describe two-dimensional shapes and the word *solid* to describe three-dimensional shapes.

Use the sides, faces and vertices of shapes to practice counting and reinforce the concept of one-to-one correspondence.

The teacher and students orally describe and name the shapes found on a **Shape Hunt**. Students draw a shape and build it using materials regularly kept in the classroom such as construction paper, clay, wooden sticks or straws.

Students can use a variety of manipulatives and real-world objects to build larger shapes with these and other smaller shapes: squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres.

Kindergarteners can manipulate cardboard shapes, paper plates, pattern blocks, tiles, canned food, and other common items.

-Focus in Kindergarten, NCTM 2011

Tools/Resources

Illustrative Mathematics tasks:

- K.G Alike or Different Game

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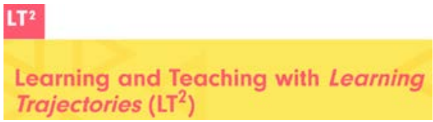

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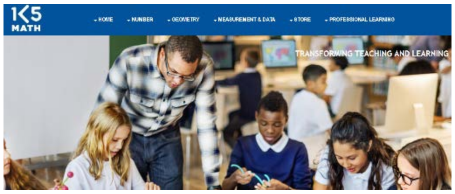

Common Misconceptions:

One of the most common misconceptions in geometry is the belief that orientation is tied to shape. Students need to have many experiences with shapes in different orientations. For example, in the *Just Two Triangles* activity, ask students to form larger triangles with the two triangles in different orientations.

It is important when students are exploring 2-dimensional shapes (flat) that the shapes they are working with are on paper or other "FLAT" material.

Domain: Geometry (G)

◆ **Cluster B:** Analyze, compare, create, and compose shapes.

Standard: K.G.5

Model shapes in the world by building shapes from components (*e.g. sticks and clay balls*) and drawing shapes. **(K.G.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected *to Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres)* in Kindergarten, and to *Reason with shapes and their attributes* in Grade 1.

Explanation and Examples:

Students are asked to apply their understanding of geometric attributes of shapes in order to create given shapes. For example, a student may roll a clump of clay into a sphere or use their finger to draw a triangle in the sand table, recalling various attributes in order to create that particular shape.

Because two-dimensional shapes are flat and three-dimensional shapes are solid, students should draw two-dimensional shapes and build three-dimensional shapes. Shapes may be built using materials such as clay, toothpicks, marshmallows, gumdrops, straws, pipe cleaners, etc.

Instructional Strategies:

See K.G.4

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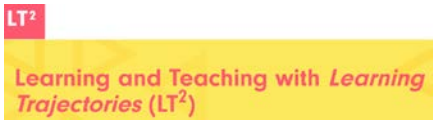

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Common Misconceptions:

See K.G.4

Domain: Geometry (G)

◆ **Cluster B:** Analyze, compare, create, and compose shapes.

Standard: K.G.6

Compose simple shapes to form larger shapes. For example, “Can you join these two triangles with full sides touching to make a rectangle?”

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections:

See K.G.1

Explanation and Examples:

This standard moves beyond identifying and classifying simple shapes to manipulating two or more shapes to create a new shape. This concept begins to develop as students first move, rotate, flip, and arrange puzzle pieces.

Next, students use their experiences with puzzles to move given shapes to make a design (e.g., “Use the 7 tangrams pieces to make a fox.”). Finally, using these previous foundational experiences, students manipulate simple shapes to make a new shape.

It is important that students are allowed to explore and build geometric understanding for themselves. One important step is to switch from making assertions and generalizations to framing ideas as questions. Rather than saying, “Every time you put two triangles together, you get a square”—a mathematically incorrect statement by the way. Ask the following: “How many different ways can you put these two triangles together to make a new shape?” “What shapes will you get?”
This allows children to see that even with two right triangles made from a square, they can put these together to make a larger triangle or a parallelogram.

Kindergartners can develop the ability to intentionally and systematically combine shapes to make new shapes and complete puzzles. They do so with increasing anticipation, on the basis of the shapes’ attributes, and thus, children developmental imagery of the component shapes. They move from using shapes separately to putting them together to make pictures.

Instructional Strategies:

Students use pattern blocks, tiles, or paper shapes and technology to make new two- and three-dimensional shapes. Their investigations allow them to determine what kinds of shapes they can join to create new shapes. They answer questions such as “What shapes can you use to make a square, rectangle, circle, triangle? …etc.”

This is an opportunity to use blocks from a play center to create shapes composed of a series of blocks. Laying several rectangular prisms can make other identifiable shapes.

Students may use a document camera to display shapes they have composed from other shapes. They may also use an interactive whiteboard to copy shapes and compose new shapes. They should describe and name the new shape.

Have students compose (build) a larger shape using only smaller shapes that have the same size and shape. The sides of the smaller shapes should touch and there should be no gaps or overlaps within the larger shape. For example, use one-inch squares to build a larger square with no gaps or overlaps.

Have students also use different shapes to form a larger shape where the sides of the smaller shapes are touching and there are no gaps or overlaps. Ask students to describe the larger shape and the shapes that formed it.

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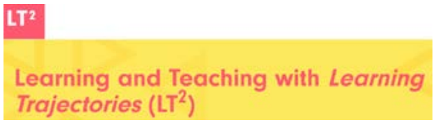
LT²

Learning and Teaching with *Learning Trajectories* (LT²)

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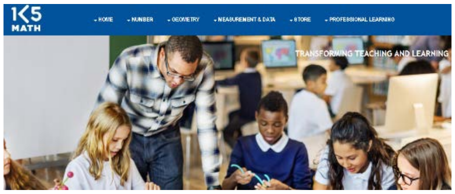


Common Misconceptions:

See K.G.4

APPENDIX: TABLE 1. Common Addition and Subtraction Situations

Shading taken from OA progression

	Result Unknown	**Change Unknown**	**Start Unknown**
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Taken from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3
	Total Unknown	**Addend Unknown**	**Both Addends Unknown[1]**
Put Together/ Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare[3]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

Blue shading indicates the four Kindergarten problem subtypes. Students in grades 1 and 2 work with all subtypes and variants (blue and green). Yellow indicates problems that are the difficult four problem subtypes or variants that students in Grade 1 work with but do not need to master until Grade 2.

[1] These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.

[2] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

[3] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.

TABLE 2. *Common Multiplication and Division Situations*

Grade level identification of introduction of problem situations taken from OA progression

	Unknown Product	**Group Size Unknown** **("How many in each group?" Division)**	**Number of Groups Unknown** **("How many groups?" Division)**
	3 x 6 = ?	3 x ? = 18; 18 ÷ 3 = ?	? x 6 = 18; 18 ÷ 6 = ?
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays[4], Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	a × b = ?	a × ? = p, and p ÷ a = ?	? × b = p, and p ÷ b = ?

Multiplicative compare problems appear first in Grade 4 (green), with whole number values and with the "times as much" language from the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit language change

TABLE 3. The Properties of Operations

the subject of the comparing sentence ("A red hat costs n times as much as the blue hat" results in the same comparison as "A blue hat is 1/n times as much as the red hat" but has a different subject.)

Name of Property	Representation of Property	Example of Property, Using Real Numbers
Properties of Addition		
Associative	$(a+b)+c=a+(b+c)$	$(78+25)+75=78+(25+75)$
Commutative	$a+b=b+a$	$2+98=98+2$
Additive Identity	$a+0=a$ *and* $0+a=a$	$9875+0=9875$
Additive Inverse	For every real number a, there is a real number $-a$ such that $a+-a=-a+a=0$	$-47+47=0$
Properties of Multiplication		
Associative	$(a\times b)\times c=a\times(b\times c)$	$(32\times5)\times2=32\times(5\times2)$
Commutative	$a\times b=b\times a$	$10\times38=38\times10$
Multiplicative Identity	$a\times1=a$ *and* $1\times a=a$	$387\times1=387$
Multiplicative Inverse	*For every real number a,* $a\neq0$, *there is a real number* $\frac{1}{a}$ *such that* $a\times\frac{1}{a}=\frac{1}{a}\times a=1$	$\frac{8}{3}\times\frac{3}{8}=1$
Distributive Property of Multiplication over Addition		
Distributive	$a\times(b+c)=a\times b+a\times c$	$7\times(50+2)=7\times50+7\times2$

(Variables a, b, and c represent real numbers.)

Excerpt from NCTM's ***Developing Essential Understanding of Algebraic Thinking****, grades 3-5 p. 16-17*

TABLE 4. The Properties of Equality

Name of Property	Representation of Property	Example of property
Reflexive Property of Equality	$a = a$	$3{,}245 = 3{,}245$
Symmetric Property of Equality	$If\ a = b, then\ b = a$	$2 + 98 = 90 + 10, then\ 90 + 10 = 2 + 98$
Transitive Property of Equality	$If\ a = b\ and\ b = c, then\ a = c$	$If\ 2 + 98 = 90 + 10\ and\ 90 + 10 = 52 + 48$ *then* $2 + 98 = 52 + 48$
Addition Property of Equality	$If\ a = b, then\ a + c = b + c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{3}{5}$
Subtraction Property of Equality	$If\ a = b, then\ a - c = b - c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} - \frac{1}{5} = \frac{2}{4} - \frac{1}{5}$
Multiplication Property of Equality	$If\ a = b, then\ a \times c = b \times c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \times \frac{1}{5} = \frac{2}{4} \times \frac{1}{5}$
Division Property of Equality	$If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \div \frac{1}{5} = \frac{2}{4} \div \frac{1}{5}$
Substitution Property of Equality	If $a = b$, then b may be substituted for *a* in any expression containing *a*.	$If\ 20 = 10 + 10$ *then* $90 + 20 = 90 + (10 + 10)$

(Variables a, b, and c can represent any number in the rational, real, or complex number systems.)

TABLE 5. The Properties of Inequality

Exactly one of the following is true: $a < b, a = b, a > b$.

$If\ a > b\ and\ b > c\ then\ a > c.$

$If\ a > b, then\ b < a.$

$If\ a > b, then - a < -b.$

$If\ a > b, then\ a \pm c > b \pm c.$

$If\ a > b\ and\ c > 0, then\ a \times c > b \times c.$

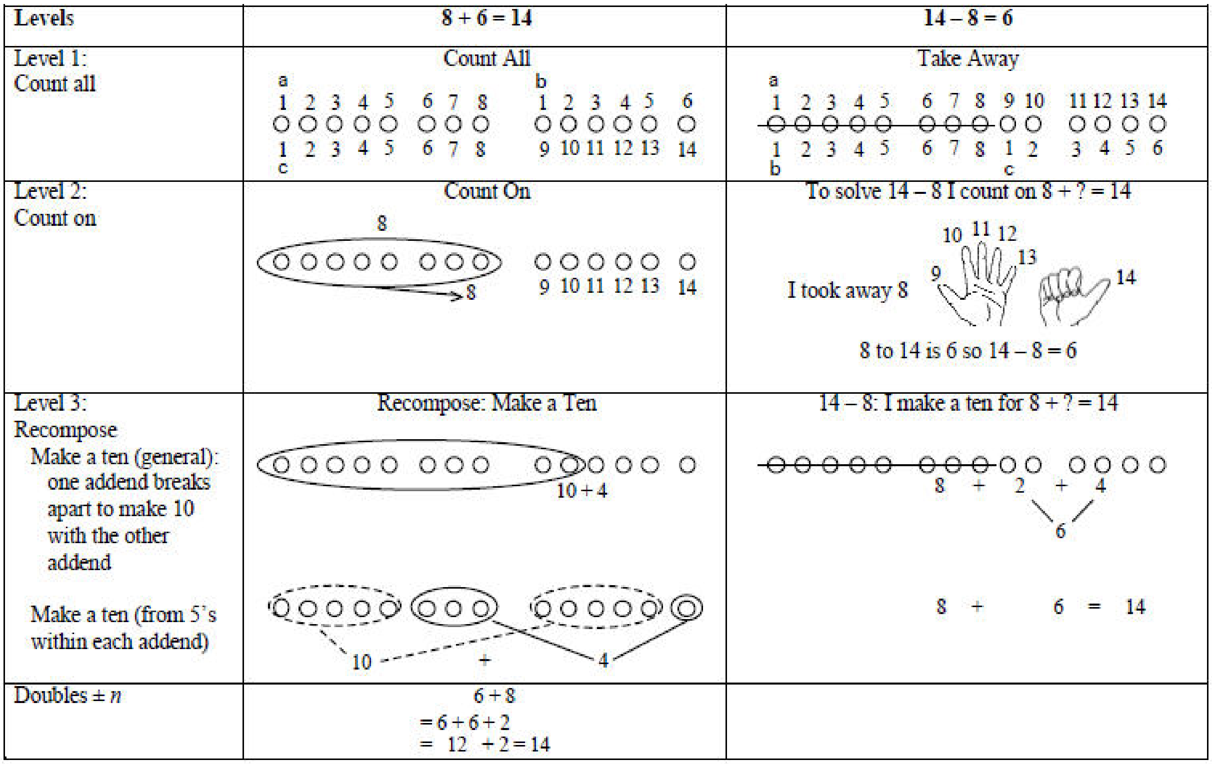
$If\ a > b\ and\ c < 0, then\ a \times c < b \times c.$

$If\ a > b\ and\ c > 0, then\ a \div c > b \div c.$

$If\ a > b\ and\ c < 0, then\ a \div c < b \div c.$

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

TABLE 6. Development of Counting in K-2 Children



Levels	8 + 6 = 14	14 – 8 = 6
Level 1: Count all	Count All a b 1 2 3 4 5 6 7 8 1 2 3 4 5 6 ○○○○○ ○○○ ○○○○○ ○ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ~~○○○○○ ○○○~~○○ ○○○○ 1 2 3 4 5 6 7 8 1 2 3 4 5 6 b c
Level 2: Count on	Count On 8 ○○○○○ ○○○ ○○○○○ ○ 8 9 10 11 12 13 14	To solve 14 – 8 I count on 8 + ? = 14 10 11 12 9 13 14 I took away 8 8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten ○○○○○ ○○○ ○○○○○ ○ 10 + 4 ○○○○○ ○○○ ○○○○○ ○ 10 + 4	14 – 8: I make a ten for 8 + ? = 14 ~~○○○○○ ○○○~~○○ ○○○○ 8 + 2 + 4 6 8 + 6 = 14
Doubles ± n	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Beginning--A child can count very small collections (1-4) collection of items and understands that the last word tells "how many" even. Beyond on small numbers the number words may be said without the lack of numerical understanding. This is often referred to as rote counting.

Level 1—The child uses one to one correspondence to find the number of objects in two sets. Even if the quantity is known for the first set, the child will start with the first set and continue counting on into the second set. The child begins the count with one. This also connects to Piaget's Hierarchical Inclusion – that in order to have 7 – you have to have 6, 5, 4, etc.

Level 2 – At this level the student begins the counting, starting with the known quantity of the first set and "counts on" from that point in the sequence to establish how many. This method is used to find the total in two sets. This counting is not rote. This level of counting requires several connections between cardinality and counting meanings of the number words.

Level 3 - At this level the student begins using known facts to solve for unknown facts. For example, the student uses "make a ten" where one addend breaks apart to make 10 with another addend OR a doubles plus/minus one strategy. Students begin to implicitly use the properties of operations.

Table 7. Cognitive Rigor Matrix/Depth of Knowledge (DOK)

Kansas Math Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom)	DOK Level 1 Recall & Reproduction	DOK Level 2 Basic Skills & Concepts	DOK Level 3 Strategic Thinking & Reasoning	DOK Level 4 Extended Thinking
Remember	• Recall conversions, terms, facts			
Understand	• Evaluate an expression • Locate points on a grid or number on number line • Solve a one-step problem • Represent math relationships in words, pictures, or symbols	• Specify, explain relationships • Make basic inferences or logical predictions from data/observations • Use models/diagrams to explain concepts • Make and explain estimates	• Use concepts to solve non-routine problems • Use supporting evidence to justify conjectures, generalize, or connect ideas • Explain reasoning when more than one response is possible • Explain phenomena in terms of concepts	• Relate mathematical concepts to other content areas, other domains • Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	• Follow simple procedures • Calculate, measure, apply a rule (e.g., rounding) • Apply algorithm or formula • Solve linear equations • Make conversions	• Select a procedure and perform it • Solve routine problem applying multiple concepts or decision points • Retrieve information to solve a problem • Translate between representations	• Design investigation for a specific purpose or research question • Use reasoning, planning, and supporting evidence • Translate between problem & symbolic notation when not a direct translation	• Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	• Retrieve information from a table or graph to answer a question • Identify a pattern/trend	• Categorize data, figures • Organize, order data • Select appropriate graph and organize & display data • Interpret data from a simple graph • Extend a pattern	• Compare information within or across data sets or texts • Analyze and draw conclusions from data, citing evidence • Generalize a pattern • Interpret data from complex graph	• Analyze multiple sources of evidence or data sets
Evaluate			• Cite evidence and develop a logical argument • Compare/contrast solution methods • Verify reasonableness	• Apply understanding in a novel way, provide argument or justification for the new application
Create	• Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	• Generate conjectures or hypotheses based on observations or prior knowledge and experience	• Develop an alternative solution • Synthesize information within one data set	• Synthesize information across multiple sources or data sets • Design a model to inform and solve a practical or abstract situation

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